ON THE WORD AND PERIOD GROWTH OF SOME
GROUPS OF TREE AUTOMORPHISMS

LAURENT BARTHOLDI AND ZORAN ŠUNIḰ

Section de Mathématiques
Université de Genève, CP 240
1211 Genève 24, Switzerland
laurent.bartholdi@math.unige.ch

Department of Mathematical Sciences
SUNY Binghamton
Binghamton, NY 13902, USA
shunka@math.binghamton.edu

ABSTRACT. We generalize a class of groups defined by Rostislav Grigorchuk in [Gri84] to a much larger class of groups, and provide upper and lower bounds for their word growth (they are all of intermediate growth) and period growth (under a small additional condition, they are periodic).

1. INTRODUCTION

Since William Burnside's original question ("do there exist finitely generated infinite groups all of whose elements have finite order?"), dozens of problems have received sometimes unexpected light from the theory of groups acting on rooted trees: to name a few, John Milnor's famous Problem 5603 [Mil68] on growth, the theory of just-infinite groups [Gri00], of groups of bounded width [BG00] where a sporadic type of group with an unusual Lie algebra was discovered, the existence of a finitely presented amenable but non-elementary amenable group [Gri99], of groups whose spectrum is a Cantor set [BG99], etc.

The main examples fall roughly in two classes, the **Grigorchuk groups** which have good combinatorial properties (like an ubiquitous "shortening lemma"), and the **GGS groups**

(named after Slava Grigorchuk, Narain Gupta and Said Sidki), which have a richer group-theoretical potential (see for instance [BG99] where torsion-free and torsion groups cohabit). We propose an extension of the class defined by R. Grigorchuk, and initiate a systematic approach of these new groups, which we propose to call **spinal groups**, since the generators are tree automorphisms that are trivial except in the neighborhood of a "spine". Our hope is that this class is

- large enough so that it remains a trove of new examples for yet-to-conceive questions, and

- small enough so that it remains amenable to quantitative analysis, in particular thanks to a "shortening lemma" that allows simple inductive proofs.

This paper is roughly comprised of two parts. The first describes these groups and the combinatorial tools required to fathom them. The second expands on estimates of the word growth and period growth for these groups.

The main tools used in the analysis of spinal groups are:

- A "shortening lemma". Each element g in a spinal group can be expressed as $g = (g_1, \ldots, g_r)h$, where h belongs to a finite group, and each of the g_i belongs to a (possibly different) spinal group. A lemma (Lemma 6.2) states that there is a norm on each group such that the sum of the norms of the g_i's is substantially less than the norm of g.

- A "portrait representation". Each element g in a spinal group can be described by a subtree $\iota(g)$ of the tree on which the group acts, with decorations on nodes of the subtree. For the flavor of portraits we use, the subtree $\iota(g)$ is finite, and its depth, size etc. carry valuable information on g.

Each spinal group G_ω is defined by an infinite sequence $\omega = \omega_1 \omega_2 \ldots$ of group epimorphisms between two fixed finite groups. When various conditions are imposed on ω, it is possible to give good bounds on the growth functions.

The spinal groups introduced in this paper differ from the Grigorchuk examples (see [Gri80], [Gri84] and [Gri85] for the Grigorchuk examples or see the description below in Subsection 3.3) in that they are groups of tree automorphisms where the degree is arbitrary (not a prime as in the Grigorchuk examples) and the root part of the group does not have to be cyclic (it does not have to be abelian either).

Just to mention a few results that are obtained:

- All spinal groups have intermediate growth, more precisely subexponential growth, and growth at least $e^{\sqrt{n}}$. Various upper bounds are provided in case the defining sequence ω shows some signs of cooperation. We consider two situations: r-homogeneous and r-factorable sequences (see Section 6), giving different bounds of the form $e^{n^{\beta}}$.

- All spinal groups defined through a regular root action are periodic. As above, upper and lower bounds for the period growth function are provided in the favorable cases; they are both polynomial (see Section 7).

- For every $\beta \in (1/2, 1)$ there exist spinal groups (even among the Grigorchuk examples) whose degree of growth γ_G is between $e^{n^{\beta}}$ and e^n, i.e. the degree satisfies $e^{n^{\beta}} \precsim \gamma_G \precsim e^n$ (see Theorem 5.3).

- There exist spinal groups (even among the Grigorchuk examples) with at least linear degree of period growth (see Theorem 7.8).

- The degree of period growth of the first Grigorchuk group is at most $n^{3/2}$ (see Theorem 7.7).

Part I. **Spinal Groups**

2. Weight Functions, Word and Period Growth

Let $S = \{s_1, \ldots, s_k\}$ be a non-empty set of symbols. A **weight function** on S is any function $\tau : S \to \mathbb{R}_{>0}$ (note that the values are strictly positive). The **weight** of any word over S is then defined by the extension of τ to a function, still written $\tau : S^* \to \mathbb{R}_{\geq 0}$, on the free monoid S^* of words over S (note that the empty word is the only word mapped to 0). Let G be an infinite group and $\rho : S^* \twoheadrightarrow G$ a surjective monoid homomorphism. (Equivalently, G is finitely generated and $\rho(S) = \{\rho(s_1), \ldots, \rho(s_k)\}$ generates G as a monoid.) The **weight** of an element g in G with respect to the triple (S, τ, ρ) is, by definition, the smallest weight of a word u in S^* that represents g, i.e. the smallest weight of a word in $\rho^{-1}(g)$. The weight of g with respect to (S, τ, ρ) is denoted by $\partial_G^{(S,\tau,\rho)}(g)$.

For n non-negative real number, the elements in G that have weight at most n with respect to (S, τ, ρ) constitute the **ball** of radius n in G with respect to (S, τ, ρ), denoted by $B_G^{(S,\tau,\rho)}(n)$. The number of elements in $B_G^{(S,\tau,\rho)}(n)$ is finite and is denoted by $\gamma_G^{(S,\tau,\rho)}(n)$. The function $\gamma_G^{(S,\tau,\rho)}$, defined on the non-negative real numbers, is called the **word growth** (or just **growth**) function of G with respect to (S, τ, ρ).

If, in addition, G is a torsion group, the following definitions also make sense. For n non-negative real number, the maximal order of an element in the ball $B_G^{(S,\tau,\rho)}(n)$ is finite

and will be denoted by $\pi_G^{(S,\tau,\rho)}(n)$. The function $\pi_G^{(S,\tau,\rho)}$, defined on the non-negative real numbers, is called the **period growth** function of G with respect to (S,τ,ρ).

A partial order $\precsim$ is defined on the set of non-decreasing functions on $\mathbb{R}_{\geq 0}$ by $f \precsim g$ if there exists a positive constant C such that $f(n) \leq g(Cn)$ for all $n \in \mathbb{R}_{\geq 0}$. An equivalence relation $\sim$ is defined by $f \sim g$ if $f \precsim g$ and $g \precsim f$. The equivalence class of $\gamma_G^{(S,\tau,\rho)}$ is called the **degree of growth** of G and it does not depend on the (finite) set S, the weight function τ defined on S and the homomorphism ρ. The equivalence class of $\pi_G^{(S,\tau)}$ is called the **degree of period growth** of G and it also does not depend on the triple (S,τ,ρ).

Of course, when we define a weight function on a group G we usually pick a finite generating subset of G closed for inversion and not containing the identity, assign a weight function to those generating elements and extend the weight function to the whole group G in a natural way, thus blurring the distinction between a word over the generating set and the element in G represented by that word and completely avoiding the discussion of ρ. In most cases everything is still clear that way.

Let us mention that the standard way to assign a weight function is to assign the weight 1 to each generator. In that case we denote the weight of a word u by $|u|$ and call it the **length** of u. In this setting, the length of the group element g is the distance from g to the identity in the Cayley graph of the group.

Since the degree of growth is invariant of the group we are more interested in it than in the actual growth function for a given generating set. For any finitely generated infinite group G, the following trichotomy exists: G is of

- **polynomial growth** if $\gamma_G(n) \precsim n^d$ for some $d \in \mathbb{N}$;
- **intermediate growth** if $n^d \precnsim \gamma_G(n) \precnsim e^n$ for all $d \in \mathbb{N}$;
- **exponential growth** if $e^n \sim \gamma_G(n)$.

We also say G is of **subexponential growth** if $\gamma_G(n) \precnsim e^n$ and of **superpolynomial growth** if $n^d \precnsim \gamma_G(n)$ for all $d \in \mathbb{N}$.

The classes of groups of polynomial and exponential growth are clearly non-empty: the former consists, by a theorem of Mikhail Gromov [Gro81], precisely of virtually nilpotent groups and the latter contains, for instance, all non-elementary hyperbolic groups [GH90].

As a consequence of Gromov's theorem, if a group G is of polynomial growth then its growth function γ_G is equivalent to n^d for an integer d. There even is a formula giving d in terms of the lower central series of G, due to Yves Guivarc'h and Hyman Bass [Gui70, Bas72].

By Tits' alternative [Tit72], there are no examples of groups of intermediate growth among the linear groups. However, R. Grigorchuk discovered examples of groups of intermediate growth by studying piecewise diffeomorphisms of the real line [Gri83], and other examples followed [FG91, Bar00a].

3. The Groups

The class of groups we are about to define is a generalization of the class of Grigorchuk p-groups introduced in [Gri80], [Gri84] and [Gri85]. An intermediate generalization was already suggested by Grigorchuk in [Gri85], but seems never to have been pursued.

Also, Alexander Rozhkov gives even more general constructions of similar type in [Roz86].

In the original description, Grigorchuk groups are given as groups of permutations of the unit interval from which a set of measure 0 is removed. In this paper we find it more convenient to describe the groups as groups of automorphisms of the q-regular rooted tree.

3.1. **Infinite regular rooted trees and tree automorphisms.** The approach we take here follows [Bri98] and [Har00].

Fix once and for all an integer $q \geq 2$, and set $Y = \{1, 2, \ldots, q\}$. We think of the q-**regular rooted tree** $\mathcal{T}^{(q)}$ as the set of finite words over Y related by the prefix ordering. Recall that a word over the alphabet Y is just a finite sequence of elements of Y; for convenience we start the indexing of the letters at 0. In the prefix ordering, $u \leq v$ if and only if u is a prefix of v, and an edge joins two vertices in $\mathcal{T}^{(q)}$ precisely when one is an immediate successor of the other.

Every finite word represents a vertex of the tree: the empty word represents the **root**, the words $1, 2, \ldots, q$ represent the vertices on the first level below the root, the two-letter words $11, 12, \ldots, 1q$ represent the vertices on the second level below the vertex 1, etc.
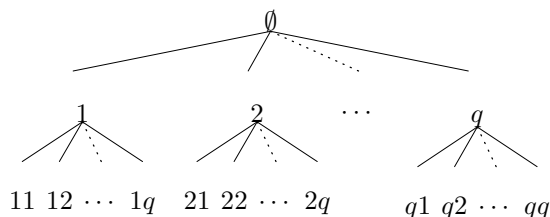
The tree $\mathcal{T}^{(q)}$

The vertex u is above the vertex v in the tree if and only if u is a prefix of v. The vertex v is a child of the vertex u if and only if $v = ui$ for some $i \in Y$. The words of length k constitute the **level** L_k in the tree.

An automorphism of the tree $\mathcal{T}^{(q)}$ is any permutation of the vertices in $\mathcal{T}^{(q)}$ preserving the prefix ordering (and therefore, also the length). Every automorphism g of $\mathcal{T}^{(q)}$ induces a permutation of the set $\partial \mathcal{T}^{(q)} = Y^{\mathbb{N}}$ of infinite sequences (again indexed from 0) over Y in a natural way. Geometrically, $\partial \mathcal{T}^{(q)}$ is the boundary of $\mathcal{T}^{(q)}$. For two infinite sequences u and v in $\partial \mathcal{T}^{(q)}$ we define $u \wedge v$ to be the longest common prefix of u and v. An automorphism g of the tree $\mathcal{T}^{(q)}$ induces a permutation $\bar{g}$ of $\partial \mathcal{T}^{(q)}$ satisfying

$$(1) \qquad\qquad |\bar{g}(u) \wedge \bar{g}(v)| = |u \wedge v|$$

for all infinite sequences u, v in $\partial \mathcal{T}^{(q)}$. Conversely, every permutation $\bar{g}$ of $\partial \mathcal{T}^{(q)}$ satisfying (1) induces an automorphism g of the tree $\mathcal{T}^{(q)}$ in a natural way.

In the sequel, it will be convenient for us to define some tree automorphisms by using this alternative way (permutations of infinite sequences). Actually, we will not distinguish between the two ways at all and we will switch back and forth between the two points of view. Also, from now on we will write $\mathcal{T}$ instead of $\mathcal{T}^{(q)}$, and will denote the automorphism group of $\mathcal{T}$ by $\mathsf{Aut}(\mathcal{T})$.

For a word u over Y denote by $\mathcal{T}_u$ the set of words in $\mathcal{T}$ that have u as a prefix. The set $\mathcal{T}_u$ has a tree structure for the prefix ordering and it is isomorphic to $\mathcal{T}$ by the canonical isomorphism deleting the prefix u. Any automorphism g of $\mathcal{T}$ that fixes the word u induces an automorphism $g_{|u}$ of $\mathcal{T}_u$ by restriction. Every automorphism $g_{|u}$ of $\mathcal{T}_u$ fixes u and induces an automorphism g_u of $\mathcal{T}$, which acts on the word w exactly as $g_{|u} \in \mathsf{Aut}(\mathcal{T}_u)$ acts on the w part of the word uw, namely by $ug_u(w) = g_{|u}(uw)$. The map φ_u defined by $g \mapsto g_{|u} \mapsto g_u$ is a surjective homomorphism from the stabilizer $\mathsf{Stab}(u)$ of u in $\mathsf{Aut}(\mathcal{T})$ to the automorphism group $\mathsf{Aut}(\mathcal{T})$.

Let $\mathsf{Stab}(L_1)$ be the stabilizer of the first level of $\mathcal{T}$ in $\mathsf{Aut}(\mathcal{T})$, i.e. $\mathsf{Stab}(L_1) = \bigcap_{i=1}^{q} \mathsf{Stab}(i)$. The homomorphism $\psi : \mathsf{Stab}(L_1) \to \Pi_{i=1}^{q} \mathsf{Aut}(\mathcal{T})$ given by

$$\psi(g) = (\varphi_1(g), \varphi_2(g), \ldots, \varphi_q(g)) = (g_1, g_2, \ldots, g_q)$$

is an isomorphism.

Similarly, let $\mathsf{Stab}(L_r)$ be the stabilizer of the r-th level of $\mathcal{T}$ in $\mathsf{Aut}(\mathcal{T})$:

$$\mathsf{Stab}(L_r) = \bigcap \{\mathsf{Stab}(u) \,|\, u \text{ is a } r\text{-letter word in } \mathcal{T}\}.$$

The homomorphism $\psi_r : \mathsf{Stab}(L_r) \to \prod_{i=1}^{q^r} \mathsf{Aut}(\mathcal{T})$ given by

$$\psi_r(g) = (\varphi_{1\ldots11}(g), \varphi_{1\ldots12}(g), \ldots, \varphi_{q\ldots qq}(g)) = (g_{1\ldots11}, g_{1\ldots12}, \ldots, g_{q\ldots qq})$$

is an isomorphism.

3.2. **The construction of the groups.** Let G_A be a group (called the **root group**) acting faithfully and transitively on Y (therefore, G_A is finite of order at least q and most $q!$). Further, let G_B be a finite group (called the **level group**) such that the set $\mathsf{Epi}(G_B, G_A)$ of surjective homomorphisms from G_B to G_A is non-empty. When an epimorphism in $\mathsf{Epi}(G_B, G_A)$ is called ω_i, denote the kernel $\mathsf{Ker}(\omega_i)$ by K_i. We impose additional requirements on G_B by asking that

- the union of all these kernels is G_B (so that every element in G_B is sent to the identity by some homomorphism in $\mathsf{Epi}(G_B, G_A)$);
- their intersection is trivial (which, among the other things, says that G_B is a subdirect product of several copies of G_A).

The set $\widehat{\Omega}$ is defined as the set of infinite sequences $\omega = \omega_1 \omega_2 \ldots$ over $\mathsf{Epi}(G_B, G_A)$ such that every non-trivial element g of G_B both appears and does not appear in infinitely many of the kernels $K_1, K_2, \ldots$. Note that the indexing of the sequences in $\widehat{\Omega}$ starts with 1. Equivalently, we might say that $\widehat{\Omega}$ consists of the sequences $\omega = \omega_1 \omega_2 \ldots$ over $\mathsf{Epi}(G_B, G_A)$ such that for every i we have

$$\bigcup_{i \leq j} K_j = G_B \qquad \text{and} \qquad \bigcap_{i \leq j} K_j = 1.$$

It is true that $\widehat{\Omega}$ depends on G_B and G_A, but we will avoid any notation emphasizing that fact. The **shift operator** $\sigma : \widehat{\Omega} \to \widehat{\Omega}$ is defined by $\sigma(\omega_1 \omega_2 \ldots) = \omega_2 \omega_3 \ldots$.

The root group G_A acts faithfully on the boundary of $\mathcal{T}$ by acting on the 0-coordinate in $Y^{\mathbb{N}} = \partial \mathcal{T}$, namely by

$$g(y_0 y_1 y_2 \ldots) = g(y_0) y_1 y_2 \ldots.$$

The automorphism of $\mathcal{T}$ induced by $g \in G_A$ will also be denoted by g and the set of non-identity automorphisms of $\mathcal{T}$ induced by G_A will be denoted by A. Letters like $a, a_1, a', \ldots$ are reserved for the elements in A.

Given a sequence ω in $\widehat{\Omega}$ we define an action of the level group G_B on the tree $\mathcal{T}$ as follows:

$$g(q \ldots q 1 y_{n+1} y_{n+2} \ldots) = q \ldots q 1 \omega_{n+1}(g)(y_{n+1}) y_{n+2} \ldots;$$
$$g(y) = y \text{ for any word } y \text{ not starting with } q \ldots q1.$$

The group G_B acts faithfully on $\mathcal{T}$ as a group of tree automorphisms. The tree automorphism corresponding to the action of g will be denoted by g_ω and the set of non-identity tree automorphisms induced by G_B will be denoted by B_ω. The abstract group G_B is canonically isomorphic to the group of tree automorphisms G_{B_ω} for any ω so that we will no make too much difference between them and will frequently omit the index ω in the notation. The index will be omitted in B_ω as well. Letters like $b, b_1, b', \ldots$ are reserved for the elements in B.

What really happens is that $a \in G_A$ acts at the root of $\mathcal{T}$ by permuting the subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_q$. On the other hand, the action of $b \in G_B$ is prescribed by ω. Namely, b acts on the subtree $\mathcal{T}_1$ exactly as $\omega_1(b) \in G_A$ would act on $\mathcal{T}$, on the subtree $\mathcal{T}_{q1}$ exactly as $\omega_2(b) \in G_A$ would act on $\mathcal{T}$, $\ldots$; and b acts trivially on subtrees not of the form $\mathcal{T}_{q\ldots q1w}$.

$\omega_1(b) \ 1 \ \cdots$

$\omega_2(b) \ 1 \ \cdots$

$\omega_3(b) \ 1 \ \ldots$
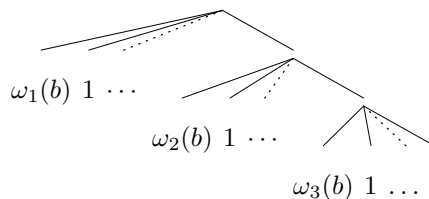
The automorphism $b_\omega \in G_B$

We define now the main object of our study:

Definition 3.1. *For any sequence $\omega \in \widehat{\Omega}$, the subgroup of the automorphism group $\mathsf{Aut}(\mathcal{T})$ of the tree $\mathcal{T}$ generated by A and B_ω is denoted by G_ω and called the **spinal group** defined by the sequence ω.*

Note that we could define a still larger class of groups if we avoided some of the self-imposed restrictions above. However, many of the properties that follow and that interest us would not hold in that larger class.

The groups G_ω, like all groups acting on a rooted tree, can be described as **automata groups** [GC71, BG99], where the elements of the group are represented by Mealy (or Moore) automata with composition of automata as group law. These automata will be finite automata if and only if the sequence ω is regular, i.e. describable by a finite automaton. We will not pursue this topic here.

3.3. **The examples of Grigorchuk.** Before we proceed with our investigation of the constructed groups let us describe exactly what groups were introduced by Grigorchuk in [Gri80], [Gri84] and [Gri85]. Each Grigorchuk p-group acts on a rooted p-regular tree, for

p a prime, the root group $G_A \cong \mathbb{Z}/p\mathbb{Z}$ is the group of cyclic permutations of $Y = \mathbb{Z}/p\mathbb{Z} = \{1, 2, \ldots, p\}$ generated by the cyclic permutation $a = (1, 2, \ldots, p)$, the level group G_B is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and only the following $p+1$ homomorphisms from G_B to G_A are used in the construction of the infinite sequences in $\widehat{\Omega}$. These epimorphisms are written $\left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]$ to mean the linear functionals on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ given by $(x, y) \mapsto ux + vy$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ p-1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The most known and investigated example is the **first Grigorchuk group** [Gri80], which is defined as above for $p = 2$ where the sequence

$$\omega = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ldots$$

is periodic of period 3.

In the case of Grigorchuk 2-groups (acting on the binary tree), it is customary to denote the only nontrivial element of the root group $G_A = \mathbb{Z}/2\mathbb{Z}$ by a and the three nontrivial elements of the level group $G_B = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by b, c and d. There are only three epimorphisms from G_B to G_A and each of them maps exactly one of the B-generators b, c, d to 1 and the other two to a. The epimorphisms sending d, c and b, respectively, to 1 are denoted by 0, 1 and 2. Then the set of admissible sequences $\widehat{\Omega}$ consists of all those sequences that contain each of these three epimorphisms infinitely many times, i.e. sequences over $\{0, 1, 2\}$ that have infinitely many appearances of each of the letters 0, 1 and 2. In this terminology, the first Grigorchuk group is defined by the sequence $012012012\ldots$.

3.4. **More examples.** It is not difficult to construct many examples where G_A and G_B are abelian, but the construction of examples where G_A and G_B are not abelian is not obvious. The following example, which allows different generalizations, was suggested by Derek Holt.

Let $G_B = \langle b_1, b_2, b_3, b_4, b_5, b_6, x_{12}, x_{34} \rangle$ where $b_1, b_2, b_3, b_4, b_5, b_6$ all have order 3 and commute with each other, x_{12} and x_{34} have order 2 and commute and

$$b_i^{x_{jk}} = \begin{cases} b_i, & \text{if } i \in \{j, k\} \\ b_i^{-1}, & \text{otherwise} \end{cases}.$$

In other words G_B is the semidirect product $(\mathbb{Z}/3\mathbb{Z})^6 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2 = \langle x_{12}, x_{34} \rangle$ and x_{12} fixes the first two coordinates of $(\mathbb{Z}/3\mathbb{Z})^6$ and acts by inversion on the last 4,

x_{34} fixes the middle 2 coordinates and acts by inversion on the other 4 and, consequently, $x_{56} = x_{12}x_{34}$ fixes the last two coordinates and inverts the first 4.

The following 12 subgroups are normal in G_B, their intersection is trivial, their union is G_B, and each factor is isomorphic to the symmetric group $\mathbb{Z}/3Z \rtimes \mathbb{Z}/2\mathbb{Z} = S_3$ which is then taken to be G_A:

$$\langle b_1, b_3, b_4, b_5, b_6, x_{12}\rangle, \qquad \langle b_1, b_2, b_3, b_5, b_6, x_{34}\rangle, \qquad \langle b_1, b_2, b_3, b_4, b_5, x_{56}\rangle,$$
$$\langle b_2, b_3, b_4, b_5, b_6, x_{12}\rangle, \qquad \langle b_1, b_2, b_4, b_5, b_6, x_{34}\rangle, \qquad \langle b_1, b_2, b_3, b_4, b_6, x_{56}\rangle,$$
$$\langle b_1 b_2, b_3, b_4, b_5, b_6, x_{12}\rangle, \quad \langle b_1, b_2, b_3 b_4, b_5, b_6, x_{34}\rangle, \quad \langle b_1, b_2, b_3, b_4, b_5 b_6, x_{56}\rangle,$$
$$\langle b_1 b_2^2, b_3, b_4, b_5, b_6, x_{12}\rangle, \quad \langle b_1, b_2, b_3 b_4^2, b_5, b_6, x_{34}\rangle, \quad \langle b_1, b_2, b_3, b_4, b_5 b_6^2, x_{56}\rangle.$$

We consider the 12 epimorphisms from G_B to G_A that are the quotient maps by these normal subgroups, and accept in $\widehat{\Omega}$ all sequences ω that uses each of these 12 homomorphisms infinitely often.

4. Some Tools for Investigation of the Groups

In this section we introduce the tools and constructions we will use in the investigation of the groups along with some basic properties that follow quickly from the given considerations.

4.1. **Triangular weights and minimal forms.** The finite set $S_\omega = A \cup B_\omega$ is the **canonical generating set** of G_ω. The generators in A are called A-**generators** and the generators in B_ω are called B-**generators**. Note that S_ω does not contain the identity and generates G_ω as a monoid, since it is closed under inversion.

A weight function τ on S will be called **triangular** if

$$\tau(a_1) + \tau(a_2) \geq \tau(a_1 a_2) \qquad \text{and} \qquad \tau(b_1) + \tau(b_2) \geq \tau(b_1 b_2),$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 a_2 \in A$ and $b_1 b_2 \in B$.

Every g in G_ω admits a minimal form with respect to a triangular weight τ

(2) $$[a_0]b_1 a_1 b_2 a_2 \ldots a_{k-1} b_k [a_k]$$

where all a_i are in A and all b_i are in B, and a_0 and a_k are optional. This is clear, since the appearance of two consecutive A-letters can be replaced either by the empty word (if the corresponding product in G_ω is trivial) or by another A-letter (if the product corresponds to a non-trivial element in G_ω). In each case the reduction of this type does not increase the weight, while it decreases the length. The same argument is valid for consecutive B-letters.

Note that the standard weight function, the length, is triangular and, therefore, admits a minimal form of type (2).

Relations of the following 4 types:

$$a_1 a_2 \rightarrow 1, \quad a_3 a_4 \rightarrow a_5, \quad b_1 b_2 \rightarrow 1, \quad b_3 b_4 \rightarrow b_5,$$

that follow from the corresponding relations in G_A and G_B for $a_i \in A$ and $b_j \in B$ are called **simple relations**. A **simple reduction** is any single application of a simple relation from left to right (indicated above by the arrows). Any word of the form (2) will be called a **reduced word** and any word can uniquely be rewritten in reduced form using simple reductions. Of course, the word and its reduced form represent the same element.

Note that the system of reductions described above is complete, i.e. it always terminates with a word in reduced form and the order in which we apply the reductions does not change the final reduced word obtained by the reduction. The second property, known as the *Church-Rosser* property, is not very important for us since we can agree to a standard way of performing the reductions (for example, always reduce at a position as close to the beginning of the word as possible).

4.2. **Some homomorphisms and subgroups.** The intersection $\mathsf{Stab}(L_1) \cap G_\omega$, denoted by H_ω, is a normal subgroup of G_ω (since $\mathsf{Stab}(L_1)$ is normal in $\mathsf{Aut}(\mathcal{T})$) and it consists of those elements of G_ω that fix the first symbol of each infinite word in $Y^{\mathbb{N}} = \partial \mathcal{T}$.

Since each element in B fixes the first level, a word u over S represents an element in H_ω if and only if the word in A-letters obtained after deleting all the B-letters in u represents the identity element.

Further, H_ω is the normal closure of B_ω in G_ω, with $G_\omega / H_\omega \cong G_A$, and H_ω is generated by the elements $b_\omega^g = g b_\omega g^{-1}$ for b in B and g in G_A.

Denote by φ_i^ω the homomorphism obtained by restricting φ_i to H_ω in the domain and to the image $\varphi_i(H_\omega)$ in the codomain and let us calculate this image. Clearly, $\psi(b_\omega) = (\omega_1(b), 1, \ldots, 1, b_{\sigma\omega})$. For any a in A, $\psi(b_\omega^a)$ has the same components as $\psi(b_\omega)$ does but in different positions depending on a. For example, if a is the cyclic permutation $(1, 2, \ldots, q)$ (meaning $1 \mapsto 2 \mapsto \cdots \mapsto q \mapsto 1$), the images of $b_\omega^{a^j}$ under various φ_i^ω are given in Table 1.

Since ω_1 is surjective and the root group acts transitively on Y we get all A and all B-generators in the image of every φ_i^ω. Therefore $\varphi_i^\omega : H_\omega \to G_{\sigma\omega}$ is a surjective homomorphism for all $i = 1, 2, \ldots, q$.

The homomorphism $\psi^\omega : H_\omega \to \prod_{i=1}^q G_{\sigma\omega}$ given by

$$\psi^\omega(g) = (\varphi_1^\omega(g), \ldots, \varphi_q^\omega(g)) = (g_1, g_2, \ldots, g_q)$$

	φ_1^ω	φ_2^ω	φ_3^ω	$\cdots$	φ_{i+1}^ω	$\cdots$	φ_{q-1}^ω	φ_q^ω
b_ω	$\omega_1(b)$	1	1	$\cdots$	1	$\cdots$	1	$b_{\sigma\omega}$
b_ω^a	$b_{\sigma\omega}$	$\omega_1(b)$	1	$\cdots$	1	$\cdots$	1	1
$b_\omega^{a^2}$	1	$b_{\sigma\omega}$	$\omega_1(b)$	$\cdots$	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$
$b_\omega^{a^i}$	1	1	1	$\cdots$	$\omega_1(b)$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b_\omega^{a^{q-2}}$	1	1	1	$\cdots$	1	$\cdots$	$\omega_1(b)$	1
$b_\omega^{a^{q-1}}$	1	1	1	$\cdots$	1	$\cdots$	$b_{\sigma\omega}$	$\omega_1(b)$

TABLE 1. The maps φ_i associated with the permutation $a = (1, 2, \ldots, q)$

is a subdirect embedding, i.e. is surjective on each factor. We will avoid the superscript ω as much as possible.

Similarly, the intersection $\mathsf{Stab}(L_r) \cap G_\omega$, denoted by $H_\omega^{(r)}$, is a normal subgroup of G_ω (since $\mathsf{Stab}(L_r)$ is normal in $\mathsf{Aut}(\mathcal{T})$) and consists of those elements of G_ω that fix the first r symbols of each infinite word in $\partial\mathcal{T}$. The homomorphism $\psi_r^\omega : H_\omega^{(r)} \to \prod_{i=1}^{q^r} G_{\sigma\omega}$ given by

$$\psi_r^\omega(g) = (\varphi_{1\ldots1}(g), \ldots, \varphi_{q\ldots q}(g)) = (g_{1\ldots1}, \ldots, q_{q\ldots q})$$

is a subdirect embedding.

We end this subsection with a few easy facts, whose proof we omit:

Lemma 4.1. *For any* $h \in H_\omega$, $g \in G_A$, $b \in B$ *and* $i \in \{1, \ldots, q\}$, *we have*

(1) $|h_i| \leq (|h| + 1)/2$.

(2) $\varphi_i(h^g) = \varphi_{g^{-1}(i)}(h)$.

(3) *The coordinates of* $\psi(b^g)$ *are:* $\omega_1(b)$ *at the coordinate* $g(1)$, b *at* $g(q)$ *and* 1 *elsewhere.*

Proposition 4.2. *The group* G_ω *is infinite for every* ω *in* $\widehat{\Omega}$.

Proof. The proper subgroup H_ω maps onto $G_{\sigma\omega}$ (for instance under φ_1^ω), so $|G_\omega| \geq |A| \cdot |G_{\sigma\omega}|$. Then the proper subgroup $H_{\sigma\omega}$ of $G_{\sigma\omega}$ maps onto $G_{\sigma^2\omega}$ (under $\varphi_1^{\sigma\omega}$), so $|G_\omega| \geq |A|^2|G_{\sigma^2\omega}|$; etc. $\square$

Proposition 4.3. *The group* G_ω *is residually finite.*

Proof. G_ω is a subgroup of $\mathsf{Aut}(\mathcal{T})$, which clearly is residually finite: it is approximated by its finite quotients given by the action on Y^n, for any $n \in \mathbb{N}$. $\qquad\square$

Proposition 4.4. *The group G_ω has a trivial center.*

Proof. First, we will prove that if $g \in G_\omega$ is central then g must be in H_ω.

Let $g = ha$ where $h \in H_\omega$ and $a \in A$. If $a(1) = 1$ and $a(i) = j$ for some $i \neq j$ then g does not commute with the elements $a' \in A$ such that $a'(1) = i$. If $a(1) \neq 1$ then choose $b \in B$ with $b \notin K_1$ and consider

$$\varphi_1([g, b]) = \varphi_1(gbg^{-1}b^{-1}) = \varphi_1(h)\varphi_{a^{-1}(1)}(b)\varphi_1(h^{-1})\omega_1(b^{-1}).$$

It is clear that $\varphi_1(h)\varphi_{a^{-1}(1)}(b)\varphi_1(h^{-1}) \in H_{\sigma\omega}$ since $a^{-1}(1) \neq 1$, which, along with the fact that $\omega_1(b^{-1}) \neq 1$, gives $\varphi_1([g, b]) \notin H_{\sigma\omega}$ and therefore $[g, b] \neq 1$.

Now, we proceed by induction on the length of the elements and we prove the statement for all ω simultaneously. From the above discussion it is clear that no A-generator and no B-generator outside of K_1 is in the center. Consider a generator $b \in K_1$ and choose an element $a \in A$ with $a(q) = 1$. Then $\varphi_1([a, b]) = \varphi_{a^{-1}(1)}(b)\omega_1(b^{-1}) = b \neq 1$. Therefore no element in K_1 is in the center and we have completed the basis of the induction.

Consider an element $g \in G_\omega$ of length ≥ 2. If $g \notin H_\omega$ we already know that g is not in the center. Let $g \in H_\omega$. At least one of the projections, say $g_i \in G_{\sigma\omega}$, is not trivial. Since g_i has strictly shorter length than g, we obtain that g_i is not in the center of $G_{\sigma\omega}$ so that g is not in the center of H_ω (and therefore not in the center of G_ω). $\qquad\square$

Proposition 4.5. *The subgroup $D_r = \langle A, K_r \rangle$ of G_ω is finite for any $r \in \mathbb{N}$.*

Proof. Take any $g \in D_r \cap H_\omega^{(r)}$, and consider the coordinates of $\psi_r^\omega(g)$. They all belong to K_r, whence $D_r \cap H_\omega^{(r)}$ is a finite group of order at most $|K_r|^{q^r}$, and D_r is finite, since $H_\omega^{(r)}$ is of finite index in G_ω. $\qquad\square$

Corollary 4.6. *For $\omega \in \widehat{\Omega}$, the subgroup $\langle A, b \rangle$ of G_ω is finite for any $b \in G_B$.*

4.3. **Tree decomposition of reduced words.** The following construction corresponds directly to a construction exhibited in [Gri84] and [Gri85]. Let

$$F = [a_0]b_1a_1b_2a_2 \ldots a_{k-1}b_k[a_k]$$

be a reduced word in S representing an element in H_ω. We rewrite the element F of G_ω in the form

$$F = b_1^{[a_0]} b_2^{[a_0]a_1} \ldots b_k^{[a_0]a_1 \ldots a_{k-1}} [a_0] a_1 \ldots a_{k-1} [a_k]$$
$$= b_1^{g_1} b_2^{g_2} \ldots b_k^{g_k} [a_0] a_1 \ldots a_{k-1} [a_k],$$

where $g_i = [a_0] a_1 \ldots a_{i-1} \in G_A$. We know that $[a_0] a_1 \ldots a_{k-1} [a_k] = 1$, since F is in H_ω. Next, using the definition of ω_1 and a table similar to Table 1 (but for all possible a) we compute the (not necessarily reduced) words $\overline{F_1}, \ldots, \overline{F_q}$ representing the elements $\varphi_1(F), \ldots, \varphi_q(F)$ of $G_{\sigma\omega}$, respectively. Then we reduce these q words using simple reductions and obtain the reduced words $F_1, \ldots, F_q$. We still have $\psi(F) = (F_1, \ldots, F_k)$. The order in which we perform the reductions is unimportant since the system of simple reductions is complete. Thus the rooted q-ary labeled tree of depth 1 whose root is decorated by the word F and its q children by the words $F_1, \ldots, F_q$ is well defined and we call it the **depth-1 decomposition of F**.

Note that each B-letter b from F contributes exactly one appearance of the letter b to one of the words $\overline{F_1}, \ldots, \overline{F_q}$ and, possibly, one A-letter to another word. Thus, the length of any of the reduced words $F_1, \ldots, F_q$ does not exceed k i.e. does not exceed $(n+1)/2$ where n is the length of F.

Given an $r > 1$ and a reduced word F representing an element in $H_\omega^{(r)}$, we construct a rooted q-ary labeled tree of depth r inductively as follows: the root is decorated by F and the decompositions of depth $r-1$ of $F_1, \ldots, F_q$ are attached to the q children of the root. We call this tree the **depth-r decomposition of F**.

Note that the vertices on the second level in the decomposition are decorated by F_{11}, F_{12} $\ldots, F_{qq}$ where F_{ij} have the property

$$\psi^{\sigma\omega}(F_i) = (\varphi_1^{\sigma\omega}(F_i), \varphi_2^{\sigma\omega}(F_i), \ldots, \varphi_q^{\sigma\omega}(F_i)) = (F_{i1}, F_{i2} \ldots, F_{iq}).$$

The vertices on the third level are decorated by $F_{111}, F_{112}, \ldots, F_{qqq}$, etc.

4.4. **The commutators.** We now determine the commutator subgroup $[G_\omega, G_\omega]$ of G_ω along with the abelianization $G_\omega^{ab} = G_\omega/[G_\omega, G_\omega]$.

For a word F over S, define the word F_B to be the B-word obtained after the removal of all A-letters in F. Then define the following set of words over S:

$$\mathsf{Ker}(\rho_B^{ab}) = \{F \in S^* \,|\, F_B \text{ represents the identity in } G_B^{ab}\}.$$

Lemma 4.7. *If F is a word over S representing identity in G_ω, then $F \in \mathsf{Ker}(\rho_B^{ab})$. In other words, all relators in G_ω come from $\mathsf{Ker}(\rho_B^{ab})$.*

We may therefore consider the map $p_B^{ab} : G_\omega \to G_B^{ab}$ given by $p_B^{ab}(g) = F_B$, where F is any word over S representing g, which is clearly surjective, and is well-defined by the lemma above:

$$
\begin{array}{ccc}
S^* & \xrightarrow{\ \rho_B^{ab}\ } & G_B^{ab} \\[2mm]
{\scriptstyle \rho_\omega}\big\downarrow & \quad\nearrow & \\[1mm]
& {\scriptstyle p_B^{ab}} & \\[1mm]
G_\omega & &
\end{array}
$$

The lemma also shows that if $g \in G_\omega$ can be represented by some word in $\mathsf{Ker}(\rho_B^{ab})$ then, for any representation $g = F$ where F is a word over S, F is in $\mathsf{Ker}(\rho_B^{ab})$. We identify $\mathsf{Ker}(\rho_B^{ab})$ with a subset of G_ω using ρ_ω, whence $\mathsf{Ker}(p_B^{ab}) = \mathsf{Ker}(\rho_B^{ab})$ and $G_\omega/\mathsf{Ker}(\rho_B^{ab}) \cong G_B^{ab}$.

Proof of lemma 4.7. The proof is by induction on the length of F and it will be done for all ω simultaneously.

The statement is clear for the empty word. Next, no word of length 1 represents the identity in any group G_ω. Now assume that the claim is true for all words of length less than n, with $n \geq 2$ and let F be a word of length n representing the identity in G_ω.

If F is not reduced, then we reduce it to a shorter word F'. Since the reduced word F' is in $\mathsf{Ker}(\rho_B^{ab})$ if and only if the original word F is in $\mathsf{Ker}(\rho_B^{ab})$ the claim follows in this case from our inductive hypothesis.

Assume F is reduced. Since F must be in H_ω the decomposition of F of depth 1 is well defined. The length of each of the (possibly not reduced) words $\overline{F_1}, \ldots, \overline{F_q}$ is at most $(n+1)/2 < n$ (since $n > 1$), so that, by the inductive hypothesis, each of the words $\overline{F_i}$ is in $\mathsf{Ker}(\rho_B^{ab})$. The set $\mathsf{Ker}(\rho_B^{ab})$ is clearly closed under concatenation, so $\overline{F_1} \cdots \overline{F_q}$ is in $\mathsf{Ker}(\rho_B^{ab})$. Assume $F_B = b_1 \cdots b_k$; then each of the B-letters b_i from F appears exactly once in some word $\overline{F_j}$, and therefore F_B and $(\overline{F_1} \cdots \overline{F_q})_B$ represent the same element in G_B^{ab}, namely, the identity. We conclude that F is in $\mathsf{Ker}(\rho_B^{ab})$. $\qquad\square$

We may define $\mathsf{Ker}(\rho_A^{ab})$ and $p_A^{ab} : G_\omega \to G_A^{ab}$ similarly. It is easy to see that any word representing the identity in G_ω must come from $\mathsf{Ker}(\rho_A^{ab})$, so $\mathsf{Ker}(p_A^{ab}) = \mathsf{Ker}(\rho_A^{ab})$ and $G_\omega/\mathsf{Ker}(\rho_A^{ab}) \cong G_A^{ab}$.

Since $G_\omega/\mathsf{Ker}(\rho_A^{ab}) \cong G_A^{ab}$ and $G_\omega/\mathsf{Ker}(\rho_B^{ab}) \cong G_B^{ab}$ are abelian, the commutator subgroup $[G_\omega, G_\omega]$ is in the intersection $\mathsf{Ker}(\rho_A^{ab}) \cap \mathsf{Ker}(\rho_B^{ab})$. On the other hand, any word F from the intersection $\mathsf{Ker}(\rho_A^{ab}) \cap \mathsf{Ker}(\rho_B^{ab})$ clearly represents the identity in the abelianization G_ω^{ab}. We have thus proved:

Theorem 4.8. *For every ω in $\widehat{\Omega}$, the commutator $[G_\omega, G_\omega]$ is equal to the intersection* $\mathsf{Ker}(\rho_A^{ab}) \cap \mathsf{Ker}(\rho_B^{ab})$. *Moreover, $G_\omega^{ab} \cong G_A^{ab} \times G_B^{ab}$.*

As a consequence, the commutator $[G_\omega, G_\omega]$ is generated, as a subgroup of G_ω, by all $[x, y]$ with $x, y \in G_A \cup G_B$.

Let us define another set of words over S:

$$\mathsf{Ker}(\rho_B) = \{F \in S^* \,|\, F_B \text{ represents the identity in } G_B\}.$$

Again, we can consider this set as a set of elements in G_ω. It is easy to see that this set is actually the normal closure $G_A^{G_\omega}$ of G_A in G_ω and it is generated as a monoid by the set $\{a^g | a \in A, g \in G_B\}$. In case G_B is abelian, the sets $\mathsf{Ker}(\rho_B^{ab})$ and $\mathsf{Ker}(\rho_B)$ clearly coincide and we have

$$G_\omega / G_A^{G_\omega} = G_\omega / \mathsf{Ker}(\rho_B) = G_\omega / \mathsf{Ker}(\rho_B^{ab}) \cong G_B^{ab} = G_B.$$

In particular, this shows that the index of the normal closure of G_A is p^2 for any Grigorchuk p-group (as defined in [Gri84] or [Gri85]).

Of course $\mathsf{Ker}(\rho_A) = \{F \in S^* \,|\, F_A \text{ represents the identity in } G_A\}$ is the normal closure of G_B in G_ω, but this is the subgroup H_ω which we already discussed.

The following result generalizes the decomposition of the first Grigorchuk group as a semidirect product. This approach allows a much more algebraic treatment of groups acting on trees.

Proposition 4.9. *Let G_A and G_B be abelian groups and suppose G_B splits as $G_B = K_1^\perp \times K_1$. Further, set $D = \langle A, K_1 \rangle$ and $T = (K_1^\perp)^{G_\omega}$, the normal closure of $K_1^\perp$ in G_ω. Then $G_\omega = T \rtimes D$.*

Moreover, the index of T in G_ω is $|D| = |G_B|^q / |G_A|^{q-1} = |K_1|^q |G_A|$.

Proof. First, note that T^{G_ω} is generated as a subgroup by the set

$$X = \{t^g, [t^g, d] \,|\, t \in K_1^\perp, g \in G_A, d \in D\}.$$

Indeed, conjugation of any generator in X by an element from G_A gives another generator, conjugation by $t \in K_1^\perp$ is unimportant since $K_1^\perp \subseteq X$ and, for $k \in K_1$,

$$k t^g k^{-1} = [k, t^g] t^g,$$

$$k [t^g, d] k^{-1} = k t^g d (t^{-1})^g d^{-1} k^{-1} = [k, t^g] t^g k d (t^{-1})^g (kd)^{-1} = [k, t^g][t^g, kd].$$

The subgroup generated by X is thus normal. On the other hand $t^g \in T$ and $[t^g, d] = t^g (t^{-1})^{dg} \in T$ so that $X \subseteq T$ and $\langle X \rangle = T$.

Since G_ω is generated by D together with the elements of the form t^g, $t \in K_1^\perp$, $g \in G_A$, we note that $G_\omega = TD$.

Let us prove that $T \cap D = 1$. Assume $g \in T \cap D$. Since $T \subseteq H_\omega$ we can consider $\psi(g) = (g_1, \ldots, g_q)$. Since $g \in D$ we have $g_i \in K_1$, for all i, and $g_i = p_B^{ab}(g_i)$. On the other hand, since $g \in T$ we have $p_B^{ab}(g_i) \in \langle K_1^\perp \rangle$, for all i. Therefore $g_i = 1$ for all i, so $g = 1$.

Consider $H_\omega \cap D$. Clearly $\psi(H_\omega \cap D) \subseteq K_1^q$. On the other hand, given $(k_1, \ldots, k_q) \in K_1^q$ we have $\psi(k_1^{g_1} k_2^{g_2} \ldots k_q^{g_q}) = (k_1, \ldots, k_q)$ where $g_i \in G_A$ with $g_i(q) = i$. Therefore $|H_\omega \cap D| = |K_1|^q$ and since the index of H_ω in G_ω is $|G_A|$ we obtain the result. $\qquad\square$

Part II. Quantitative Estimates

In the following sections we will impose various restrictions on the sequence ω defining the group G_ω and give estimates of word and period growth in those cases. All the estimates will be done with respect to the canonical generating set $S_\omega = A \cup B_\omega$. As a shorthand, we will use $\gamma_\omega(n)$ and $\pi_\omega(n)$ instead of $\gamma_{G_\omega}(n)$ and $\pi_{G_\omega}(n)$.

5. The Word Growth in the General Case

A finite subsequence $\omega_{i+1}\omega_{i+2}\ldots\omega_{i+r}$ of a sequence ω in $\widehat{\Omega}$ is **complete** if each element of G_B is sent to the identity by at least one homomorphism from the sequence $\omega_{i+1}\omega_{i+2}\ldots\omega_{i+r}$, i.e. if $\bigcup_{j=1}^r \mathsf{Ker}(\omega_{i+j}) = G_B$. We note that a complete sequence must have length at least $q+1$ since all the kernels have index $|G_A| \geq q$ in G_B. In particular, the length of a complete sequence is never shorter than 3. Note that by definition all sequences in $\widehat{\Omega}$ can be factored into finite complete subsequences.

Theorem 5.1. *G_ω has subexponential growth, for all ω in $\widehat{\Omega}$.*

Let F be a reduced word of length n representing an element in $H_\omega^{(r)}$ and consider the decomposition of the word F of depth r. For $\ell = 0, \ldots, r$, define the length $|L_\ell(F)|$ of the level ℓ to be the sum of the lengths of the elements on the level ℓ. The following lemma is a direct generalization of Lemma 1 in [Gri85]. The proof is similar, but adapted to the more general setting of the present paper.

Lemma 5.2 (3/4-Shortening)**.** *Let $\omega \in \widehat{\Omega}$ be a sequence that starts with a complete sequence of length r. Then the following inequality holds for every reduced word F representing an element in $H_\omega^{(r)}$:*

$$|L_r(F)| \leq \frac{3}{4}|F| + q^r.$$

Proof. Define ξ_i to be the number of B-letters from $K_i - (K_{i-1} \cup \cdots \cup K_1)$ appearing in the words at the level $i - 1$ and ν_i to be the number of simple reductions performed to get the words $F_{j_1 \ldots j_i}$ on the level i from their unreduced versions $\overline{F_{j_1 \ldots j_i}}$.

A reduced word F of length n has at most $(n+1)/2$ B-letters. Every B-letter in F that is in K_1 contributes one B-letter and no A-letters to the unreduced words $\overline{F_1}, \ldots, \overline{F_q}$. The B-letters in F that are not in K_1 (there are at most $(n+1)/2 - \xi_1$ such letters) contribute one B-letter and one A-letter. Finally, the ν_1 simple reductions reduce the number of letters on level 1 by at least ν_1. Therefore,

$$|L_1(F)| \leq 2((n+1)/2 - \xi_1) + \xi_1 - \nu_1 = n + 1 - \xi_1 - \nu_1.$$

In the same manner, each of the ξ_2 B-letters on level 1 that is from $K_2 - K_1$ contributes at most one B-letter to the words on level 2 and the other B-letters (at most $(|L_1(F)| + q)/2 - \xi_2$ of them) contribute at most 2 letters, so

$$|L_2(F)| \leq n + 1 + q - \xi_1 - \xi_2 - \nu_1 - \nu_2.$$

Proceeding in the same manner, we obtain the estimate

(3) $\qquad |L_r(F)| \leq n + 1 + q + \ldots q^{r-1} - \xi_1 - \xi_2 - \cdots - \xi_r - \nu_1 - \nu_2 - \cdots - \nu_r.$

If $\nu_1 + \nu_2 + \cdots + \nu_r \geq n/4$, then the claim of the lemma follows. Assume therefore

(4) $\qquad\qquad\qquad\qquad \nu_1 + \nu_2 + \cdots + \nu_r < n/4.$

For $i = 0, \ldots, r - 1$, define $|L_i(F)|^+$ to be the number of B-letters from $B - (K_1 \cup \cdots \cup K_i)$ appearing in the words at the level i. Clearly, $|L_0(F)|^+$ is the number of B-letters in F and

$$|L_0(F)|^+ \geq \frac{n-1}{2}.$$

Going from the level 0 to the level 1, each B-letter contributes one B letter of the same type. Thus, the words $\overline{F_1}, \ldots, \overline{F_q}$ from the first level before the reduction takes place have exactly $|L_0(F)|^+ - \xi_1$ letters that come from $B - K_1$. Since we lose at most $2\nu_1$ letters due to the simple reductions, we obtain

$$|L_1(F)|^+ \geq \frac{n-1}{2} - \xi_1 - 2\nu_1.$$

Next we go from level 1 to level 2. There are $|L_1(F)|^+$ B-letters on level 1 that come from $B - K_1$, so there are exactly $|L_1(F)|^+ - \xi_2$ B-letters from $B - (K_1 \cup K_2)$ in the words $\overline{F_{11}}, \ldots, \overline{F_{qq}}$ and then we lose at most $2\nu_2$ B-letters due to the reductions. We get

$$|L_2(F)|^+ \geq \frac{n-1}{2} - \xi_1 - \xi_2 - 2\nu_1 - 2\nu_2,$$

and, by proceeding in a similar manner,

$$(5) \qquad |L_{r-1}(F)|^+ \geq \frac{n-1}{2} - \xi_1 - \ldots \xi_{r-1} - 2\nu_1 - \ldots 2\nu_{r-1}.$$

Since $\omega_1 \ldots \omega_r$ is complete, we have $\xi_r = |L_{r-1}(F)|^+$ and the inequalities (3), (4) and (5) give

$$|L_r(F)| \leq \frac{n}{2} + \frac{1}{2} + 1 + q + \ldots q^{r-1} + \nu_1 + \cdots + \nu_{r-1} - \nu_r.$$

which implies our claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We finish the proof of Theorem 5.1 using either the argument given in [Gri85] or the one in [Har00, Theorem VIII.61]; namely let

$$e_\omega = \limsup_{n\to\infty} \sqrt[n]{\gamma_\omega(n)}$$

denote the **exponential growth rate** of G_ω. It is known that this rate is 1 if and only if the group in question has subexponential growth. By the previous lemma we have $e_\omega \leq e_{\sigma^r\omega}^{3/4}$ and since the e_ω are bounded (for instance, by $|A \cup B|$), it follows that $e_\omega = 1$ for all $\omega \in \widehat{\Omega}$.

5.1. **A lower bound for word growth.** A general lower bound, tending to e^n when $q \to \infty$, exists on the word growth, and holds for all spinal groups:

Theorem 5.3. G_ω has superpolynomial growth, for all ω in $\widehat{\Omega}$. Moreover, the growth of G_ω satisfies

$$e^{n^\alpha} \precsim \gamma_\omega(n),$$

where $\alpha = \frac{\log(q)}{\log(q) - \log\frac{1}{2}}$.

Proof. Let $\gamma_\omega = \gamma_\omega^{(S,|\cdot|,\rho)}$ denote the growth of G_ω with respect to word length. We will obtain

$$(6) \qquad \gamma_\omega(2qn + K_\omega) \geq L_\omega \gamma_{\sigma\omega}(n)^q \text{ for all } n \in \mathbb{R}_{\geq 0}$$

for some positive constants $K_\omega, L_\omega \geq 0$. Then, x applications of (6) yield, neglecting the (unimportant) constant K_ω, $\gamma_\omega((2q)^x) \geq L_\omega^{1+q+\cdots+q^{x-1}} \gamma_{\sigma^x\omega}(1)^{q^x}$, from which the theorem's claim follows. See [Bar00b, Corollary 9] for a similar proof.

We now prove (6). Choose some $h \in A$ with $h(1) = q$, let ν be a (set) retraction $A \to G_B$ of ω_1, and consider the (set) map

$$\lambda : \begin{cases} A \ni a & \mapsto h\nu(a)h^{-1} = \nu(a)^h \\ B \ni b & \mapsto b. \end{cases}$$

defined on reduced words over $A \cup B$. Note that λ does not in general induce a group homomorphism, though this is the case for the first Grigorchuk group, where it is traditionally called σ. We may, however, naturally consider $\lambda(F) \in H_\omega$.

Given any reduced word F representing $x \in G_{\sigma\omega}$, we obtain an element $y = \lambda(F)$ of H_ω, that has the following properties:

$$\varphi_q^\omega(y) = x;$$

$$\text{if } h(q) \neq 1, \text{ then } \varphi_1^\omega(y) \in G_A \text{ and } \varphi_{h(q)}^\omega(y) \in \nu(W),$$

$$\text{where } W \text{ is the set of } A\text{-letters in } x;$$

$$\text{if } h(q) = 1, \text{ then } \varphi_1^\omega(y) \in \langle A, \nu(W) \rangle;$$

$$\varphi_i^\omega(y) = 1 \text{ for all } i \notin \{1, q, h(q)\}.$$

In case $h(q) = 1$, we restrict our consideration to words F such that at most q of their A-letters are not in $\langle h \rangle$. It then follows in all cases that all coordinates except the q-th of $\lambda(F)$ are bounded. To prove the only non-trivial case, suppose $h(q) = 1$ and $F = F_0 a_1 F_1 \ldots a_q F_q$, where the F_i are words over $\langle h \rangle \cup B$. Then

$$\varphi_1^\omega(\lambda(F)) = \alpha_0 \nu(a_1) \alpha_1 \ldots \nu(a_q) \alpha_q,$$

where $\alpha_i \in \langle A, \nu(h) \rangle$. By Corollary 4.6, this last group is finite, so each α_i is bounded (say of length at most M); then $\varphi_1^\omega(\lambda(F))$ is bounded, of length at most $N = q + (q+1)M$.

Given words $F_1, \ldots, F_q$ each of length at most n, we wish to construct a word F with $\psi^\omega(F) = F'(F_1, \ldots, F_q)F''$ where F' and F'' belong to a finite set $\mathcal{F}$. Let us choose elements a_i, for $i \in Y$, such that $a_i(q) = i$. We take

$$F = \lambda(F_1)^{a_1} \ldots \lambda(F_{q-1})^{a_{q-1}} \lambda(F_q).$$

In every coordinate i, we get F_i (as $\varphi_i^\omega(\lambda(F_i)^{a_i})$), and other words, each of which is bounded. We may therefore take for $\mathcal{F}$ the set of words of length at most $(q-1)N$.

Note now that for any word F_i we have $|\lambda(F_i)| \leq 2|F_i| + 1$, so $|F| \leq q(2n+1) + q$. Also, F has at most q A-letters not in $\langle h \rangle$, namely $a_1, a_1^{-1} a_2, \ldots, a_{q-1}^{-1}$. Finally, F determines $F_1, \ldots, F_q$ up to the choice of F' and F'', so (6) holds with $K = 2q$ and $L = |\mathcal{F}|^{-2}$ $\quad\square$

Note that if q is a prime power p^α, the group G_ω is residually-p so has growth at least $e^{\sqrt{n}}$ by [Gri89]. The previous result is an improvement in all cases but $q = 2$.

For the special case of the first Grigorchuk group slightly better results exist, due to Yuriĭ Leonov [Leo98] who obtained $\gamma(n) \succsim e^{n^{0.5041}}$, and to the first author [Bar00b] who obtained

$\gamma(n) \succsim e^{n^{0.5157}}$. There is no doubt that a similar improvement of Theorem 5.3 for general spinal groups is possible.

6. THE WORD GROWTH IN THE CASE OF HOMOGENEOUS SEQUENCES

A sequence ω in $\widehat{\Omega}$ is r-**homogeneous** (for $r \geq 3$) if every finite subsequence of length r is complete. The set of r-homogeneous sequences in $\widehat{\Omega}$ will be denoted by $\Omega^{(r)}$. Note that $\Omega^{(r)}$ is closed under the shift σ, a fact that is crucial for the arguments that follow, but we will not mention it explicitly anymore. Implicitly, all sequences ω in this section will come from $\Omega^{(r)}$ for some fixed r. We will prove the following:

Theorem 6.1 (η-Estimate). *If ω is an r-homogeneous sequence, then the growth function of the group G_ω satisfies*

$$\gamma_\omega(n) \precsim e^{n^\alpha}$$

where $\alpha = \frac{\log(q)}{\log(q) - \log(\eta_r)}$ and η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$.

6.1. **A triangular weight function on G_ω.** The following weight assignment generalizes the approach taken in [Bar98] by the first author in order to estimate the growth of the first Grigorchuk group.

Consider the linear system of equations in the variables $\tau_0, \ldots, \tau_r$:

(7)
$$\begin{cases} \eta_r(\tau_0 + \tau_i) = \tau_0 + \tau_{i-1} & \text{for } i = r, \ldots, 2, \\ \eta_r(\tau_0 + \tau_1) = \tau_r. \end{cases}$$

The solution is given, up to a constant multiple, by

(8)
$$\begin{cases} \tau_i = \eta_r^r + \eta_r^{r-i} - 1 & \text{for } i = r, \ldots, 1, \\ \tau_0 = 1 - \eta_r^r. \end{cases}$$

If we also require $\tau_1 + \tau_2 = \tau_r$ we get that η_r must be a root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$. If we choose η_r to be the root of this polynomial that is between 0 and 1 we obtain that the solution (8) of the system (7) satisfies the additional properties

(9) $$0 < \tau_1 < \cdots < \tau_r < 1, \qquad 0 < \tau_0 < 1,$$

(10) $$\tau_i + \tau_j \geq \tau_k \text{ for all } 1 \leq i, j, k \leq r \text{ with } i \neq j.$$

The index r in η_r will be sometimes omitted without warning.

Now, given $\omega \in \Omega^{(r)}$, we define the weight of the generating elements in S_ω as follows: $\tau(a) = \tau_0$, for a in A and $\tau(b_\omega) = \tau_i$, where i is the smallest index with $\omega_i(b) = 1$.

Clearly, τ is a triangular weight function. The only point worth mentioning is that if b and c are two B-letters of the same weight and $bc = d \neq 1$ then d has no greater weight than b or c (this holds because $b_\omega, c_\omega \in \mathsf{Ker}(\omega_i)$ implies $d_\omega \in \mathsf{Ker}(\omega_i)$).

For obvious reasons, the weight $\partial_{G_\omega}^{(S_\omega, \tau, \rho)}(g)$ for $g \in G_\omega$ will be denoted by $\partial^\tau(g)$ and, more often, just by $\partial(g)$.

6.2. **A tree representation of the elements in G_ω.** Let g be an element in G_ω. There is a unique element h in G_A such that hg is in H_ω. We extend the map ψ to G_ω by $\psi(g) = \psi(hg)$ and we write $\psi(g) = (g_1, \ldots, g_q)$ in this case. This notation does not interfere (too much) with our previous agreement since $h = 1$ for g in H_ω. Note that the extended ψ is not a homomorphism (nor is it injective: it is $|G_A|$-to-one).

Lemma 6.2 (η-Shortening). *Let $g \in G_\omega$. Then*

$$\sum_{i=1}^{q} \partial^\tau(g_i) \leq \eta_r\big(\partial^\tau(g) + \tau_0\big).$$

Proof. Let a minimal form of g be

$$g = [a_0]b_1 a_1 \ldots b_{k-1} a_{k-1} b_k [a_k].$$

Then hg can be written in the form $hg = h[a_0]b_1 a_1 \ldots a_{k-1} b_k [a_k]$ and rewritten in the form

(11) $$hg = b_1^{g_1} \ldots b_k^{g_k},$$

where $g_i = h[a_0]a_1 \ldots a_{i-1} \in G_A$. Clearly, $\partial(g) \geq (k-1)\tau_0 + \sum_{j=1}^{k} \tau(b_j)$, which yields

(12) $$\sum_{i=1}^{k} \eta(\tau_0 + \tau(b_j)) \leq \eta(\partial(g) + \tau_0).$$

Now, observe that if the B-generator b is of weight τ_i with $i > 1$ then $\psi(b^g)$ has as components one B-generator of weight τ_{i-1} and one A-generator (of weight τ_0 of course) with the rest of the components trivial. Thus, such a b^g (from (11)) contributes at most $\tau_0 + \tau_{i-1} = \eta(\tau_0 + \tau(b))$ to the sum $\sum \partial(g_i)$. On the other hand, if b is a B-generator of weight τ_1 then $\psi(b^g)$ has as components one B-generator of weight at most τ_r, and the rest of the components are trivial. Such a b^g contributes at most $\tau_r = \eta(\tau_0 + \tau(b))$ to the sum $\sum \partial(g_i)$. Therefore

(13) $$\sum_{i=1}^{q} \partial^\tau(g_i) \leq \sum_{j=1}^{k} \eta(\tau_0 + \tau(b_j))$$

and the claim of the lemma follows by combining (12) and (13). $\qquad\square$

A simple corollary of the lemma above is that for any ζ with $\eta < \zeta < 1$ there exists a positive constant $K_\zeta = \eta\tau_0/(\zeta - \eta)$ such that $\sum_{i=1}^q \partial(g_i) \leq \zeta\partial(g)$ for every g in G_ω with $\partial(g) \geq K_\zeta$, and, therefore,

$$(14) \qquad\qquad\qquad\qquad \partial(g_i) \leq \zeta\partial(g)$$

for all $i = 1, \ldots, q$ and every g in G_ω with $\partial(g) \geq K_\zeta$. For the discussion that follows just pick a fixed value for ζ.

Starting with an element g in G_ω we construct a rooted, q-regular, labeled tree $\iota(g)$ each of whose leaves is decorated by an element of weight $\leq K$ (for a chosen $K \geq K_\zeta$) and each of whose interior vertices is decorated by an element of G_A. Technically, ι depends on K but we choose not to indicate this in our notation.

The tree $\iota(g)$ is called the **portrait** of g of size K and is constructed inductively as follows: if the weight of g is $\leq K$ then the portrait of g is the tree that has one vertex decorated by g; if $\partial(g) > K$, then the weight of all g_i's is at most $\zeta\partial(g)$ (see (14)) and the portrait of g is the tree that has h at its root and the trees $\iota(g_1), \ldots, \iota(g_q)$ attached at the branches below the root.

The map ι sending each g in G_ω to its portrait is injective (see the corresponding proof in [Bar98]). The main points are that ψ is injective on H_ω and for every g in G_ω the element $h \in G_A$ such that hg is in H_ω is unique.

Lemma 6.3. *There exists a positive constant K such that*

$$L(n) \precsim n^\alpha$$

with $\alpha = \frac{\log(q)}{\log(q)-\log(\eta_r)}$, where $L(n)$ is the maximal possible number of leaves in the portrait of size K of an element of weight at most n.

Proof. Let $\kappa = \eta\tau_0/(q - \eta)$. Choose K so that $K \geq \max\left(q^{1/\alpha} + \kappa, K_\zeta\right)$ is big enough in order that

$$j + \left(\frac{q}{q-j}\right)^{\alpha-1}\left(n - \kappa + \frac{\kappa j}{\eta}\right)^\alpha \leq (n - \kappa)^\alpha$$

be satisfied for all $n > K$ and all $j = 0, \ldots, q - 1$. Such a choice is possible because $(q/(q-j))^{\alpha-1} < 1$ for $j = 1, \ldots, q - 1$ and the two expressions are equal for $j = 0$.

Define a function $L'(n)$ on $\mathbb{R}_{>0}$ by

$$L'(n) = \begin{cases} 1 & \text{if } n \leq K, \\ (n - \kappa)^\alpha & \text{if } n > K. \end{cases}$$

We prove, by induction on n, that $L(n) \leq L'(n)$. If the weight of g is $\leq K$, the portrait has 1 leaf and $L'(n) = 1$. Otherwise, the portrait of g is made up of those of $g_1, \ldots, g_q$. Let the weights of these q elements be $n_1, \ldots, n_q$. By Lemma 14 we have $n_i \leq \zeta n$, so by induction the number of leaves in the portrait of g_i is at most $L'(n_i)$, $i = 1, \ldots, q$ and the number of leaves in the portrait of g is, therefore, at most $\sum_{i=1}^{q} L'(n_i)$.

Suppose that j of the numbers $n_1, \ldots, n_q$ are no greater than K and the other $q - j$ are greater than K, where $0 \leq j \leq q - 1$. Without loss of generality we may assume $n_1, \ldots, n_j \leq K < n_{j+1}, \ldots, n_q$. Using Jensen's inequality, Lemma 6.2, the fact that $\eta^{\alpha} = q^{\alpha-1}$ and $0 < \alpha < 1$, and direct calculation, we see that

$$
\sum_{i=1}^{q} L'(n_i) = j + \sum_{i=j+1}^{q} (n_i - \kappa)^{\alpha} \leq j + (q - j) \left(\frac{1}{q-j} \sum_{i=j+1}^{q} (n_i - \kappa) \right)^{\alpha}
$$
$$
= j + \left(\frac{q}{q-j} \right)^{\alpha-1} \left(n - \kappa + \frac{\kappa j}{\eta} \right)^{\alpha} \leq (n - \kappa)^{\alpha} = L'(n),
$$

where the last used inequality holds by the choice of K.

In case none of $n_1, \ldots, n_q$ is greater than K, we have $\sum_{i=1}^{q} L'(n_i) = q$ which is no greater than $(n - \kappa)^{\alpha} = L'(n)$, again, by the choice of K. $\qquad\square$

Lemmata 4.1 and 6.2 can be summed up in a

Scholium. The portrait of an element g of weight n is a tree of sublinear size n^{α} and logarithmic depth $\log_2(n)$.

6.3. **Proof of Theorem 6.1 (η-Estimate).** The number of elements in G_{ω} of weight at most n, i.e. the number of elements in $B_{\omega}(n)$ is equal to the number of trees in $\iota(B_{\omega}(n))$. This number is bounded above by the number of labeled, rooted, q-regular trees with at most $L(n)$ leaves where each of the leaves is decorated by an element of weight at most K and each interior vertex by an element of G_A.

The number $N(m)$ of labeled, rooted, q-regular trees with exactly m interior vertices (and, therefore, exactly $(q - 1)m + 1$ leaves) is $\frac{1}{qm+1} \binom{qm+1}{m} \sim e^m$ (see [GS95, page 1033]). Thus the number $D(m)$ of such trees with at most m interior vertices is also $\sim e^m$. A tree with at most $L(n)$ leaves has at most $I(n) = (L(n) - 1)/(q - 1) \sim n^{\alpha}$ interior vertices, so that the number of labeled trees we are interested in is $\sim e^{n^{\alpha}}$.

The decoration of the interior vertices can be done in at most $|G_A|^{I(n)} \sim e^{n^{\alpha}}$ ways.

Finally, note that different leaves of a tree representation live on different levels and therefore in different groups $G_{\sigma^t \omega}$. But the number of elements of weight at most K is bounded above by a finite number, denote it by $\gamma_f(K)$, for all groups on at most $|A \cup B|$

generators. Thus the decoration of the leaves can be chosen in at most $\gamma_f(K)^{L(n)} \sim e^{n^\alpha}$ ways.

Therefore $\gamma_\omega(n) \precsim e^{n^\alpha}$ and the proof is complete.

6.4. **Word growth in the case of factorable sequences.** We say that a sequence in $\widehat{\Omega}$ is r**-factorable** if it can be factored in complete subsequences of length at most r. The set of r-factorable sequences will be denoted by $\Omega^{[r]}$. Clearly, $\Omega^{(r)} \subset \Omega^{[r]} \subset \Omega^{(2r-1)}$ and the inclusions are proper. An upper bound on the degree of word growth can thus be obtained from Theorem 6.1, but we can do slightly better if we combine Lemma 5.2 with the idea of portrait of an element.

Theorem 6.4 (3/4-Estimate). *If ω is an r-factorable sequence, then the growth function of the group G_ω satisfies*

$$\gamma_\omega(n) \precsim e^{n^\alpha}$$

where $\alpha = \frac{\log(q^r)}{\log(q^r) - \log(3/4)} = \frac{\log(q)}{\log(q) - \log\left(\sqrt[r]{3/4}\right)}$.

Proof. Let ω be an r-factorable sequence, factored in complete words of lengths $r_1, r_2, r_3, \ldots$, with all $r_i \leq r$. We can define a modification of the portrait of an element by requiring that whenever we "blow up" a leaf on the level $r_1 + r_2 + \cdots + r_{i-1}$ because its size is too big, we expand it r_i levels down (i.e. the original word is expanded r_1 levels down, the words on the level r_1 are expanded r_2 levels, the words on the level $r_1 + r_2$ are expanded r_3 levels, etc.) and obtain $q^{r_i} \leq q^r$ new leaves. An analog of Lemma 5.2 still holds, i.e. if the length of g is at most n then sum of the lengths of the elements at the newly obtained q^{r_i} leaves is at most $\frac{3}{4}n + O(1)$ (indeed, the fact that we multiply by an element from G_A here and there does not increase the B-length of the words in question, so that the sum of the lengths at the newly obtained leaves is still at most $\frac{3}{4}n$ plus a constant). Proceeding as before completes the proof of the theorem. $\qquad\square$

The 3/4-Estimate is obtained only for the class of Grigorchuk p-groups defined by r-homogeneous (not r-factorable as above) sequences by Roman Muchnik and Igor Pak in [MP99] by different means. On the other hand, their approach gives slightly better results for the Grigorchuk 2-groups defined by r-homogeneous sequences except in the case $r = 3$ in which case the estimates coincide.

We can provide a small improvement in a special case that includes all Grigorchuk 2-groups. Namely, we are going to assume that ω is an r-factorable sequence such that each factor contains three homomorphisms whose kernels cover G_B. Note that this is possible

only when $q = 2$. Also, note that in case $q = 2$ we must have $G_A = \mathbb{Z}/2\mathbb{Z}$ since that is the only group that acts transitively and faithfully on the two-element set $Y = \{1, 2\}$. Since G_B is a subdirect product of several copies of G_A we must have $G_B = (\mathbb{Z}/2\mathbb{Z})^d$ for some $d \geq 2$, i.e. G_B is an elementary abelian 2-group.

Lemma 6.5 (2/3-Shortening). *Let $q = 2$ and $\omega \in \widehat{\Omega}$ be a sequence such that there exist 3 letters ω_k, ω_ℓ and ω_m, $1 \leq k < \ell < m \leq r$, with the property that $K_k \cup K_\ell \cup K_m = G_B$. Then the following inequality holds for every reduced word F representing an element in $H_\omega^{(r)}$:*

$$|L_r(F)| < \frac{2}{3}|F| + \frac{2}{3} + 3q^r.$$

Proof. For $i = 0, \ldots, r$, denote by $|L_i|_A$, $|L_i|_B$ and $|L_i|_{K_j}$ the number of A-letters, B-letters and B-letters from K_j, respectively, in the words on the level i of the decomposition of depth r of F.

Clearly, $|L_0|_B \geq |L_1|_B \geq \cdots \geq |L_r|_B$. Also,

$$|L_{i+1}|_A \leq |L_i|_B - |L_i|_{K_{i+1}},$$

since every B-letter from the level i contributes at most one A-letter to the next level, except for those B-letters that are in K_{i+1}. This gives

$$|L_{i+1}|_B \leq |L_{i+1}|_A + q^{i+1} \leq |L_i|_B - |L_i|_{K_{i+1}} + q^{i+1},$$

and therefore

$$|L_r|_B \leq |L_{k-1}|_B - |L_{k-1}|_{K_k} + q^k - |L_{\ell-1}|_{K_\ell} + q^\ell - |L_{m-1}|_{K_m} + q^m$$

(15)
$$< (|L_{k-1}|_{K_\ell} - |L_{\ell-1}|_{K_\ell}) + (|L_{k-1}|_{K_m}) - |L_{m-1}|_{K_m} + 3q^r,$$

since $|L_{k-1}|_B \leq |L_{k-1}|_{K_k} + |L_{k-1}|_{K_\ell} + |L_{k-1}|_{K_m}$.

It is easy to see that

$$|L_i|_{K_j} - |L_{i+1}|_{K_j} \leq |L_i|_B - |L_{i+1}|_B.$$

Indeed, any change (up or down) in the number of letters in K_j going from the level i to the level $i + 1$ is due to simple reductions involving letters from K_j, but each such simple reduction also changes (always down) the total number of B-letters by the same amount. Then, by telescoping,

(16)
$$|L_i|_{K_j} - |L_t|_{K_j} \leq |L_i|_B - |L_t|_B,$$

whenever $i \leq t$.

Combining (15) and (16) gives

$$3|L_r|_B \leq |L_r|_B + |L_{\ell-1}|_B + |L_{m-1}|_B$$

$$< (|L_{k-1}|_B - |L_{\ell-1}|_B) + (|L_{k-1}|_B - |L_{m-1}|_B) + 3q^r + |L_{\ell-1}|_B + |L_{m-1}|_B$$

$$\leq 2|L_{k-1}|_B + 3q^r \leq 2|L_0|_B + 3q^r,$$

which implies our result since $|L_r(F)| \leq 2|L_r|_B + q^r$ and $|L_0|_B \leq (|F| + 1)/2$. $\quad\square$

Let us mention here that, for every k, there exists a reduced word F of length $n = 24k$ in the first Grigorchuk group representing an element in $H_\omega^{(3)}$ such that $|L_3(F)| = 16k = 2n/3$. An example of such a word is $(abadac)^{4k}$. Thus, the lemma above cannot be improved in the sense that there cannot be an improvement unless one starts paying attention to reductions that are not simple. Of course, $(abadac)^{16} = 1$ in the first Grigorchuk group, so by introducing other relations the multiplicative constant of Lemma 6.5 could be sharpened.

As a corollary to the shortening lemma above, we obtain:

Theorem 6.6 (2/3-Estimate). *Let $q = 2$ and ω be an r-factorable sequence such that each factor contains three letters whose kernels cover G_B. Then the growth function of the group G_ω satisfies*

$$\gamma_\omega(n) \precsim e^{n^\alpha}$$

where $\alpha = \frac{\log(q^r)}{\log(q^r) - \log(2/3)} = \frac{\log(q)}{\log(q) - \log\left(\sqrt[r]{2/3}\right)}$.

The previous theorem and the lemma just before could be generalized to the other values of q, but the shortest possible complete sequence in those cases would be at least 4 and at best we would obtain the 3/4-Estimate already provided before.

6.5. **Calculations and comparisons.** The Table 2 included below lists various values of α (always rounded up), for different q and r, such that $\gamma_\omega(n) \precsim e^{n^\alpha}$. The entries in the last column indicate conditions on ω. A row labeled by "homo." indicates that the estimate is valid for r-homogeneous sequences. The estimate in a row labeled by "fact." is valid for r-factorable sequences. In addition, the labels "η", "3/4" and "2/3" indicate which estimate is used.

Note that the η-estimate for an r-homogeneous sequence is always better than the 2/3-Estimate or 3/4-Estimate for an r-factorable sequence (since $\eta_r < \sqrt[r]{2/3} < \sqrt[r]{3/4}$).

Also, note that the best available 3/4-Estimate is sometimes better than the best available η-Estimate. There are sequences ω is in $\Omega^{[r]}$ with the property that the smallest value of s such that ω is s-homogeneous is $2r - 1$. The 3/4-Estimate for r-factorable sequences is

q	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	Condition
2	.768	.836	.872	.896	.912	.924	.933	.940	homo. (η)
2	.837	.873	.896	.912	.923	.932	.939	.945	fact. $(2/3)$
2	.879	.906	.924	.936	.945	.951	.956	.961	fact. $(3/4)$
3		.890	.916	.932	.943	.951	.957	.961	homo. (η)
3		.939	.951	.959	.964	.969	.972	.975	fact. $(3/4)$
4			.932	.945	.954	.960	.965	.969	homo. (η)
4			.960	.967	.972	.975	.977	.980	fact. $(3/4)$
5				.952	.960	.966	.970	.973	homo. (η)
5				.972	.976	.979	.981	.983	fact. $(3/4)$

TABLE 2. Comparison of the obtained estimates

better than the η-Estimate for $(2r-1)$-homogeneous sequences. For the sake of an example, take $q=3$ and $r=4$ and let $\omega = 0123321001233210\ldots$ where all homomorphisms $0,1,2,3$ are required to form a complete sequence. Then, the smallest value of s such that ω is s-homogeneous is $s=7$ and the $3/4$-Estimate gives $\alpha = .939$ while the η-Estimate gives $\alpha = .943$.

7. PERIODICITY AND PERIOD GROWTH IN CASE OF A REGULAR ACTION

¿From now on we assume, in addition to the earlier conditions, that the action of G_A on Y is regular, i.e. each permutation of Y induced by an element in G_A is regular (recall a permutation is **regular** if all of its cycles have the same length). The number of elements in G_A must be equal to $|Y| = q$ in case of a faithful and regular action. Also, note that if G_A is abelian it must act regularly on Y.

The group order of an element g will be denoted by $\pi(g)$. In case F is a word $\pi(F)$ denotes the order of the element represented by the word F.

7.1. **Period decomposition and periodicity.** We will describe a step in a procedure introduced in [Gri84] that will help us to determine some upper bounds on the period growth of the constructed groups.

Let F be a reduced word of even length of the form

$$(17) \qquad F = b_1 a_1 \ldots b_k a_k.$$

Rewrite F in the form $F = b_1 b_2^{g_2} \ldots b_k^{g_k} a_1 \ldots a_k$, where $g_i = a_1 \ldots a_{i-1}$, $i = 2, \ldots, k$. Set $g = a_1 \ldots a_k \in G_A$ and let its order be s, a divisor of q. Note that $g = 1$ corresponds to

$F \in H_\omega$. The length of each cycle of g is s because of the regularity of the action. Put $H = b_1 b_2^{g_2} \dots b_k^{g_k}$ and consider the element $F^s = (Hg)^s \in H_\omega$. We rewrite this element in the form $F^s = H H^g \dots H^{g^{s-1}}$ and then in the form

$$(18) \qquad F^s = (b_1 b_2^{g_2} \dots b_k^{g_k})(b_1^g (b_2^{g_2})^g \dots (b_k^{g_k})^g) \dots \left(b_1^{g^{s-1}} (b_2^{g_2})^{g^{s-1}} \dots (b_k^{g_k})^{g^{s-1}} \right).$$

Next, by using tables similar to Table 1 (but for all possible a), we calculate the (possibly unreduced) words $\overline{F_1}, \dots, \overline{F_q}$ representing $\varphi_1(F^s), \dots, \varphi_q(F^s)$, respectively, and then we use simple reductions to get the reduced words $F_1, \dots, F_q$. The tree that has F at its root and $F_1, \dots, F_q$ as leaves on the first level is called the **period decomposition** of F. Clearly $\psi(F^s) = (F_1, \dots, F_q)$ holds and the order $\pi(F)$ of F is divisor of $q \cdot \gcd(\pi(F_1), \dots, \pi(F_q))$ since ψ is injective, s divides q and the elements $(F_1, 1, \dots, 1), \dots, (1, \dots, 1, F_q)$ commute in $\Pi_{i=1}^{q} G_{\sigma\omega}$.

The notation introduced above for the vertices in the period decomposition interferes with the notation introduced before for the vertices in the decomposition of words, but we are not going to use the latter anymore.

Let us make a couple of simple observations on the structure of the possibly unreduced words $\overline{F_1}, \dots, \overline{F_q}$ used to obtain the reduced words $F_1, \dots, F_q$ of the period decomposition.

The conjugate elements $b_1, b_1^g, \dots, b_1^{g^{s-1}}$ appear in the expression (18). The generator b_1 contributes exactly one appearance of the letter b_1 to $\overline{F_q}$. The other conjugates $b_1^g, \dots, b_1^{g^{s-1}}$ of b_1 contribute exactly one appearance of the letter b_1 to the words $\overline{F_{g(q)}}, \dots, \overline{F_{g^{s-1}(q)}}$, respectively. Similarly, $b_i^{g_i}$ contributes exactly one appearance of the letter b_i to the word $\overline{F_{g_i(q)}}$ and each of its conjugates $(b_i^{g_i})^g, \dots, (b_i^{g_i})^{g^{s-1}}$ contributes exactly one appearance of b_i to the words $\overline{F_{gg_i(q)}}, \dots, \overline{F_{g^{s-1}g_i(q)}}$. Since the length of each g-orbit is s we see that, as far as the B-letters are concerned, no word $\overline{F_i}$ gets more than one of each of the letters $b_1, \dots, b_k$, possibly not in that order. Similarly, no word $\overline{F_i}$ can get more than k A-letters and it is possible to get k A-letters only if none of the letters $b_1, \dots, b_k$ is in K_1. More precisely, the maximal number of A-letters in any $\overline{F_i}$ is $k - |F|_{K_1}$.

Theorem 7.1. *Let ω be a sequence in $\widehat{\Omega}$. Then the group G_ω is periodic.*

Proof. We will prove that the order of any element g in G_ω divides some power of q. The proof is by induction on the length n of g and it will be done for all ω simultaneously.

The statement is clear for $n = 0$ and $n = 1$. Assume that it is true for all words of length less than n, where $n \geq 2$, and consider an element g of length n.

If n is odd the element g is conjugate to an element of smaller length and we are done by the inductive hypothesis. Assume then that n is even. Clearly, g is conjugate to an element

that can be represented by a word of the form

$$F = b_1 a_1 \ldots b_k a_k.$$

In this case $\pi(g) = \pi(F)$ divides $q \cdot \gcd(\pi(F_1), \ldots, \pi(F_q))$ and if all the words F_i have length shorter than n we are done by the inductive hypothesis.

Assume that some of the words F_i have length n. This is possible only when F does not have any B-letters from K_1. Also, the words $\overline{F_i}$ corresponding to the words F_i of length n must be reduced, so that the words F_i having length n have the same B-letters as F does. For each of these finitely many words we repeat the discussion above; namely, for each such F_i of length n we construct the period decomposition. Either all of the constructed words F_{ij} are strictly shorter than n, and we get the result by induction; or some have length n, but the B-letters appearing in them do not come from $K_1 \cup K_2$.

This procedure cannot go on forever since $K_1 \cup K_2 \cup \ldots K_r = G_B$ holds for some $r \in \mathbb{N}$. Therefore at some stage we get a shortening in all the words and we conclude that the order of F is a divisor of some power of q. $\qquad\square$

7.2. **Period shadow and period growth in case of homogeneous sequences.** We can give a polynomial upper bound on the period growth of G_ω in case ω is a homogeneous sequence. In order to do so, we will make another use of the triangular weight function τ introduced before.

Lemma 7.2. *Let* $F = b_1 a_1 b_2 \ldots b_k a_k$ *be a reduced word of length* $2k$. *Then*

(19) $$\tau(F_i) \le \eta_r \tau(F), \quad \text{for all} \quad 1 \le i \le q.$$

Proof. $\tau(F) = \sum_{i=1}^k (\tau_0 + \tau(b_i))$, yielding $\sum_{i=1}^k \eta(\tau_0 + \tau(b_i)) = \eta \tau(F)$. Using an argument similar to that in Lemma 6.2 and the observations on the structure of the words $\overline{F_i}$ given above, we conclude that

$$\tau(F_i) \le \sum_{i=1}^k \eta(\tau_0 + \tau(b_i)) = \eta \tau(F).$$

$\qquad\square$

Note that all the canonical generators of G_ω have weight no more than 1. Given an element g in G_ω we construct a rooted, q-regular, labeled tree, whose leaves are decorated by elements of weight at most 1 and whose interior vertices are decorated by divisors of q. We call such a tree $\mathrm{m}(g)$ a **period shadow** of g (of size 1). Note that $\mathrm{m}(g)$ is not uniquely determined by g — nor does it uniquely determine g.

A period shadow of g is constructed inductively as follows: let g' be an element of minimal weight in the conjugacy class of g. If $\tau(g') \leq 1$ then the shadow is the tree with one vertex decorated by g'; if $\tau(g') > 1$, we assume g' is represented by a word F in the form (17), from which we construct words $F_1, \ldots, F_q$ each of weight at most $\eta\partial(F)$ and group order s (the order of $a_1 \ldots a_k$) dividing q. A shadow of g is the tree with s at its root and $\mathrm{m}(F_1), \ldots, \mathrm{m}(F_q)$ attached to the root.

Let C be the gcd of the periods of all the elements of G_ω with weight at most 1. If $\mathrm{m}(g)$ is a shadow of g, then

(20)
$$\pi(g) \text{ divides } Cq^d,$$

where d is the **depth** of $\mathrm{m}(g)$, i.e. the length of the longest path from the root to a leaf. If g is an element of G_ω of weight n, the depth of the shadow of g cannot be greater than $\lceil \log_{1/\eta}(n) \rceil$, so the following theorem holds:

Theorem 7.3 (Period η-Estimate). *If ω is an r-homogeneous sequence, then the period growth function of the group G_ω satisfies*

$$\pi_\omega(n) \precsim n^{\log_{1/\eta_r}(q)}$$

where η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$.

In a similar manner we can prove the following two theorems.

Theorem 7.4 (Period 3/4-Estimate). *If ω is an r-factorable sequence, then the period growth function of the group G_ω satisfies*

$$\pi_\omega(n) \precsim n^{r \log_{4/3}(q)}.$$

Instead of a proof, let us just note that in the process of building a shadow of size 1 of an element g of ordinary length n we are not sure that there is a shortening in the length at each level, but there is a shortening by at least a factor of $3/4$ after no more than r levels. Thus, the depth of such a shadow cannot be greater than $r\lceil \log_{4/3}(n) \rceil$ and the claim follows.

Theorem 7.5 (Period 2/3-Estimate). *If $q = 2$ and ω is an r-factorable sequence such that each factor contains three letters whose kernels cover G_B, then the period growth function of the group G_ω satisfies*

$$\pi_\omega(n) \precsim n^{r \log_{3/2}(q)}.$$

7.3. **Period growth in the case of a prime degree.** In addition to the regularity requirement we assume that the degree q of the tree $\mathcal{T}$ is a prime number. Thus the root group G_A is cyclic of prime order $q = p$ and there is no loss in generality if we assume that G_A is generated by the cyclic permutation $a = (12\ldots p)$. We assume all this in this subsection without further notice.

Let us describe the construction of a sequence that we call the **period sequence** of an element g in G_ω.

First we represent g by a reduced word F_g. Then we conjugate F_g until we get either a word F of length 1 in which case we stop, or we get a cyclically reduced word F of the form (17). This word either represents an element in H_ω in which case we stop or it has to be raised to the p-th power to get an element in H_ω. Consider the latter case and take a look again at the expression (18). Clearly,

$$\varphi_1(F^p) = \varphi_1(H)\varphi_1(H^g)\ldots\varphi_1(H^{g^{p-1}}) = \varphi_1(H)\varphi_{g^{-1}(1)}(H)\ldots\varphi_{g^{1-p}(1)}(H),$$

where $H = b_1 b_2^{g_2} \ldots b_k^{g_k} \in H_\omega$. Similarly,

$$\varphi_i(F^p) = \varphi_i(H)\varphi_i(H^g)\ldots\varphi_i(H^{g^{p-1}}) = \varphi_i(H)\varphi_{g^{-1}(i)}(H)\ldots\varphi_{g^{1-p}(i)}(H),$$

so that all the elements F_i from the period decomposition are conjugate and we have $\pi(g) = \pi(F) = p\pi(F_1)$.

Each of the letters $b_1, \ldots, b_k$ appears in $\overline{F_1}$. Also, each of the A-letters or identity factors $\omega_1(b_1), \ldots, \omega_1(b_k)$ appears in $\overline{F_1}$. Thus, if $b = b_1 b_2 \ldots b_k$ in G_B we have $b_\omega = \rho_B(F)$ and $\rho_B(\overline{F_1}) = \rho_B(F_1) = b_{\sigma\omega}$, because G_B is commutative. We may also write $b = \rho_B(F) = \rho_B(\overline{F_1}) = \rho_B(F_1)$, by dropping the indices as usual.

On the other hand, we have $\rho_A(\overline{F_1}) = \rho_A(F_1) = \omega_1(b_1)\ldots\omega_1(b_k) = \omega_1(b)$.

We conjugate the word F_1 until we get a word of length 1 or a word of the form (17). The conjugation does not change the projections $\rho_B(F_1) = b$ and $\rho_A(F_1) = \omega_1(b)$ and can only decrease the length. Now, the cyclically reduced version of F_1 has the same order as F_1 and either represents an element in H_ω or it has to be raised to the p-th power to get an element in H_ω. In the first case we stop. In the latter case we construct, as before, a word F_{11} such that $\pi(g) = \pi(F) = p\pi(F_1) = p^2\pi(F_{11})$, $\rho_B(F_{11}) = b$ and $\rho_A(F_{11}) = \omega_2(b)$.

This process cannot last forever, since $\omega_i(b) = 1$ holds for some i. The sequence $F, F_1, \ldots, F_{\underbrace{1\ldots1}_{t}}$ obtained this way has the property that the last word in the sequence, denoted F', has length 1 or represents an element in $H_{\sigma^t\omega}$. Also $\pi(F) = p^t\pi(F')$.

Theorem 7.6. *Let $q = p$ be a prime and ω an r-homogeneous word.*

If $p \geq 3$ or $p = 2$ and each subsequence of ω of length r contains three homomorphisms whose kernels cover G_B, then the period growth function of the regular spinal group G_ω satisfies

$$\pi_\omega(n) \precsim n^{(r-1)\log_2(p)}.$$

Proof. As usual, we use induction on n and we prove the statement simultaneously for all r-homogeneous ω. We will prove that $\pi_\omega(n) \leq C p^{(r-1)\log_2(n)}$ where $C = p^2$.

The statement is obvious for $n = 1$.

Consider an element g of length n, $n \geq 2$ and let $F, F_1, \ldots, F_{\underbrace{1\ldots1}_t} = F'$ be its period sequence. We know that $t \leq r$ because the word ω is r-homogeneous.

If F' has length 1 then it has order p and $\pi(g) = p^{t+1} \leq p^{r+1} = C p^{r-1} \leq C p^{(r-1)\log_2(n)}$.

Consider the case when F' has (even) length greater than 1 and $F_t \in H_{\sigma^t\omega}$. In that case $\pi(F') = \pi(W)$ for some W in $G_{\sigma^{t+1}\omega}$ that has length at most half the length of F' (the word W is one of the leaves of the period decomposition of F').

In case $t \leq r-1$ we have $\pi(g) \leq p^{r-1}\pi_{\sigma^r\omega}(n/2)$ which is no greater than $C p^{(r-1)\log_2(n)}$ by the induction hypothesis.

Let $t = r$ and $p \geq 3$. Since $t = r$ we know that the length of F' is at most $3n/4$ so that the length of W is at most $3n/8$ and we have $\pi(g) \leq p^r \pi_{\sigma^{r+1}\omega}(3n/8)$ which is no greater than $C p^{(r-1)\log_2(n)}$ by the induction hypothesis and the fact that $r \geq 4$ holds in this case.

In case $t = r$ and $p = 2$, the length of F' is at most $2n/3$ and we have $\pi(g) \leq p^r \pi_{\sigma^{r+1}\omega}(n/3)$ which is no greater than $C p^{(r-1)\log_2(n)}$ by the induction hypothesis and the fact that $r \geq 3$ holds in this case.

Therefore, in each case $\pi(g) \leq C p^{(r-1)\log_2(n)} = C n^{(r-1)\log_2(p)}$, which proves our claim. □

Note that the theorem above gives the estimate $\pi_\omega(n) \precsim n^{r-1}$ in case $q = p = 2$ and every subsequence of ω of length r contains 3 homomorphisms whose kernels cover G_B. In case this last condition does not hold we can still give the estimate $\pi_\omega(n) \precsim n^r$.

7.4. **Period growth for Grigorchuk 2-groups.** We give here a tighter upper bound on the period growth of the Grigorchuk 2-groups. It is based on a more precise observation of the process described in the previous subsection.

Theorem 7.7. *Let G_ω be a Grigorchuk 2-group. If ω is an r-homogeneous word, then the period growth function of the group G_ω satisfies*

$$\pi_\omega(n) \precsim n^{r/2}.$$

Proof. Let $\chi_\omega : G_\omega \to G_\omega^{ab}$ be the abelianization map. Recall that $G_\omega^{ab} = \langle a \rangle \times \langle b, c \rangle$ is the elementary 2-group of rank 3. We recast the construction of the period sequence as follows: in the graph below, nodes correspond to images of elements g under χ_*; arrows indicate taking a projection, φ_1 or φ_2. Double arrows indicate a squaring was applied before taking the projection (because g was not yet in H_*). Also, recall that in the squaring case the obtained projections $\varphi_1(g^2)$ and $\varphi_2(g^2)$ are conjugate. A condition labeling an edge indicates that such an edge can exist only if the condition is satisfied.
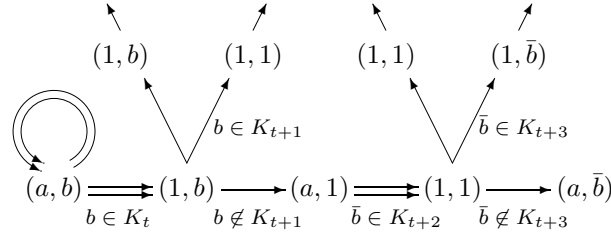
One step of the algorithm for calculating the order of an element

We proved in the previous section that all double arrows are as described. Let us complete the proof for single arrows. According to Theorem 4.8, the commutator of a Grigorchuk group is

$$[G_\omega, G_\omega] = \langle x = [a,b], y = [a,c], z = [a,d] \rangle.$$

Let us take an arbitrary element $g \in G_\omega$ with $\chi(g) = (a,b)$ and assume that b is in the kernel K_t, but it is not in any kernel with smaller index. We follow our squaring procedure $t-1$ times obtaining an element $f \in G_{\sigma^{t-1}\omega}$ and set $h = \varphi_2(f^2)$. We know that $\pi(g) = 2^t\pi(h)$, $\chi(h) = (1,b)$ and we wish to compute $\chi(\varphi_i(h))$ for $i = 1, 2$. For this purpose, write $f = u_1 \ldots u_m ab$ for some $u_i \in \{x, y, z\}$. Then

$$h = \varphi(f^2) = \varphi_2(u_1)\ldots\varphi_2(u_m)\varphi_2(u_1^a)\ldots\varphi_2(u_m^a)b.$$

Now, note that $x^a = x^{-1}$, $y^a = y^{-1}$, $z^a = z^{-1}$ and $\varphi_2(x) = \varphi_2(x^a) = b$, $\varphi_2(y) = ca$, $\varphi_2(x^a) = ac$, $\varphi_2(y) = da$, $\varphi_2(x^a) = ad$, so that all B-letters appear in pairs in the expression

(21) $$E = \varphi_2(u_1)\ldots\varphi_2(u_m)\varphi_2(u_1^a)\ldots\varphi_2(u_m^a).$$

Of course, the element represented by E is in $H_{\sigma^t\omega}$ and can be rewritten in the form $b_1^{g_1}\ldots b_k^{g_k}$, where $b_i \in \{b, c, d\}$ and $g_i \in \{1, a\}$. Let $X_o(E)$ and $X_e(E)$ denote the product of the B-letters in E preceded by an odd and even number, respectively, of a's. Those B-letters preceded by odd number of a's will appear conjugated by a when we rewrite E in the

form $b_1^{g_1} \ldots b_k^{g_k}$ and those preceded by even number of a's will appear without conjugation. It is not difficult to see that we have either $X_o(E) = X_e(E) = 1$ or $X_o(E) = X_e(E) = b$. Indeed, if the number of a's in the expression $\varphi_2(u_1) \ldots \varphi_2(u_m)$ is odd, i.e, $\varphi_2(u_1) \ldots \varphi_2(u_m)$ is not in H then both the number of c factors and the number of d factors in both $X_o(E)$ and $X_e(E)$ are even so they cancel out and the number of b factors in $X_o(E)$ and $X_e(E)$ is equal, so that their product is 1 or b. Similarly, if the number of a's in the expression $\varphi_2(u_1) \ldots \varphi_2(u_m)$ is even then the number of b factors in both $X_o(E)$ and $X_e(E)$ is even so the b's cancel out and the number of c factors and d factors in $X_o(E)$ and $X_e(E)$ is equal and even so their product is 1 or b.

Considering the extra b in the expression for h we may suppose, up to a permutation of the indices o and e, that $X_o(h) = b$ and $X_e(h) = 1$. Then

$$\chi(\varphi_1(h)) = (1, b),$$

$$\chi(\varphi_2(h)) = \begin{cases} (a, 1), & \text{if } b \notin K_{t+1} \\ (1, 1), & \text{if } b \in K_{t+1} \end{cases}$$

The same argument works when we start with $\bar{f} \in [G_{\sigma^{t+1}\omega}, G_{\sigma^{t+1}\omega}]a$ and set $\bar{h} = \varphi_2(\bar{f}^2)$; we then obtain either $X_o = X_e = 1$ whence $\chi(\varphi_i(\bar{h})) = (1, 1)$ for $i = 1, 2$, or $X_o = X_e = \bar{b}$, where $\bar{b}$ is the only B-letter in the kernel K_{t+2}, whence

$$\chi(\varphi_i(\bar{h})) = \begin{cases} (a, \bar{b}), & \text{if } \bar{b} \notin K_{t+3} \\ (1, \bar{b}), & \text{if } \bar{b} \in K_{t+3} \end{cases} \quad \text{for } i \in \{1, 2\}.$$

Thus, we enter the graph above at the vertex (a, b), loop $t - 1$ times at (a, b), then in step t move to the vertex $(1, b)$ and either use two consecutive single arrows to exit the graph and lend into an element of $G_{\sigma^{t+2}\omega}$ or we proceed to the vertex $(a, 1)$, follow another double (squaring) arrow to $(1, 1)$ and then either leave the graph through two consecutive single arrows and lend into an element of $G_{\sigma^{t+4}\omega}$ or we lend in an element in $G_{\sigma^{t+3}\omega}$ with projection $(a, \bar{b})$. The order of g will depend on the exit point, i.e. on the number of squarings performed and the various length reductions that occurred during the trip through the graph. Recall that each time we follow a non-squaring arrow we may claim a length reduction by a factor of $1/2$ and in case we follow r consecutive squaring arrows (i.e. $t = r$) we may claim an additional length reduction by a factor of $2/3$.

As usual, we can now use induction on n and prove the statement simultaneously for all r-homogeneous ω (just like in the proof of Theorem 7.6). $\qquad\square$

Roughly speaking the previous theorem says that the ratio between the number of squarings and the number of halvings performed to calculate the order of an element does not exceed $r/2$, i.e. in the worst case each r squarings are accompanied by at least 2 halvings. Theorem 7.6 from the previous subsection states, more moderately, that in the worst case each $r - 1$ raisings to the p-th power are accompanied by at least one halving step.

7.5. **Lower bounds on period growth.** In this subsection we present a construction of words of "large" order and "small" length, thus providing a lower bound on the period growth of some regular spinal groups.

The construction in the (proof of the) next theorem generalizes an unpublished idea of Igor Lysionok, who constructed short words of high order in the first Grigorchuk group.

Even though the words we construct are far from optimal, they give a polynomial lower bound on the degree of period growth in the considered cases. More precisely, we show:

Theorem 7.8. *Let ω be a word in $\widehat{\Omega}$ and $a \in G_A$ be of order 2 and satisfy $a(1) = q$. For all $j \in \mathbb{N}$, set $K_{j,a} = \{b \in B \mid \omega_j(b) = a\}$ and*

$$I = \{1\} \cup \{i > 1 \mid \omega_1 = \omega_i \text{ and } \omega_{i-1} \neq \omega_i\}.$$

Assume that I is infinite, the difference between two consecutive indices in I is at most r, $K_{1,a} \cap K_{j,a} \neq \emptyset$ for all $j > 1$ and $K_1 \cap K_{j-1,a} \neq \emptyset$ for $j \in I, j > 1$. Then

$$n^{1/(r-1)} \precsim \pi_\omega(n).$$

Proof. Assume that $s + 1$ lies in I and that $G_{\sigma^s \omega}$ contains an element g satisfying the following conditions:

(1) g is of order M;
(2) g has a representation of length $2k$ with k odd;
(3) this representation is of the form $ab_1 ab_2 \ldots ab_k$, with all the b_i in $K_{s+1,a}$ except for one, which is in K_{s+1}.

We shall construct an element g' of G_ω of order at least $2M$, having a representation of length $2(2^{s-1}k + 1)$ satisfying Condition 3 (with 1 instead of $s + 1$).

We restrict our attention to words of the form $a * a * \ldots a*$, with $* \in B$. A **word-set** is such a word, but where the $*$'s are non-empty subsets of B. An **instance** of a word-set is a word (or group element) obtained by choosing an element in each set.

For all $i \in \mathbb{N}$, there is a map from word-sets in $G_{\sigma^i \omega}$ to word-sets in $H_{\sigma^{i-1} \omega}$, defined as follows:

$$\beta_i(b) = b \text{ for all } b \subseteq B;$$

$$\beta_i(a) = aK_{i,a}a.$$

Any instance h' of $\beta_i(h)$ satisfies $\psi(h') = (*, \ldots, *, h'')$ where h'' is an instance of h.

Let us now consider $g \in G_{\sigma^s \omega}$ of order M. Set first $g'' = \beta_1 \beta_2 \ldots \beta_s(g)$ and note that $\psi_s(g'') = (g, *, \ldots, *)$, where each $*$ represents an element in $G_{\sigma^s \omega}$ and we are not interested in their actual value.

Choose $x_1, x_2 \in K_{1,a}$ such that $x_1 x_2 = b_i$, where b_i is the only letter from K_1 in g'' and replace b_i by the word-set $x_1 a K_1 a x_2$. Note that $K_{i,a}^2 = K_i$ so that the choice indicated in the previous sentence can be done. Also note that this transformation does not change the first coordinate of $\psi_s(g'')$. We claim that g'' has an instance that, as a word, is a square, say of g'. Then since $\psi_s((g')^2) = (*, \ldots, *, g)$, we will have constructed a word g' satisfying the required conditions.

Let us compute the lengths. Before the substitution of the element in K_1, g'' is a word-set of length $2^{s+1} k$. The substitution of the element in K_1 increases the length by 4. Thus, the length of the word set g'' is $2^{s+1} k + 4$. Also, the number of appearances of K_1 in g'' is 1.

Write g'' in 2 lines of length $2^s k + 2$. Each line will be of the form $a * \ldots a *$, with the $*$'s elements or subsets of B. Our goal is to choose an instance of g'' such that the two lines are identical, i.e, we want to choose identical elements in each column. Half of the columns will consist of a's and the other $2^{s-1} k + 1$ columns (an odd number of them!) will have one of the following:

(1) $K_{1,a}$ and $K_{j,a}$;
(2) $K_{1,a}$ and $b_i \in K_{s+1,a} = K_{1,a}$ ($k - 1$ columns);
(3) $K_{1,a}$, and x_i (two columns);
(4) K_1 and $K_{s,a}$ (one column).

In each case the two elements in the column can be chosen identical.

The above construction is a single step in an inductive construction in which, starting with an element $ab \in G_{\sigma^{t-1} \omega}$ where $t \in I$, $b \in K_1 = K_t$, x repetitions of the step give an element in G_ω of order at least 2^{x+2} and length at most $2^{\frac{2^{(r-1)(x+1)}-1}{2^{r-1}-1}}$, so that $n^{\frac{1}{r-1}} \precsim \pi_\omega(n)$. $\qquad \square$

Just for the sake of illustration, let us consider an example. Assume $\omega = \omega_1 \omega_2 \omega_3 \omega_1 \ldots$ with $\omega_1 \neq \omega_3$ and start with the element $g = ab_1 ab_2 ad \in G_{\sigma^3 \omega}$ where $d \in K_4 = K_1$ and

$b_1, b_2 \in K_{4,a} = K_{1,a}$. Let $x_1 x_2 = d$ where $x_1, x_2 \in K_{1,a}$. We have

$$\beta_1 \beta_2 \beta_3(g) = \beta_1 \beta_2 (aK_{3,a}ab_1aK_{3,a}ab_2aK_{3,a}ad) =$$

$$= \beta_1(aK_{2,a}aK_{3,a}aK_{2,a}ab_1aK_{2,a}aK_{3,a}aK_{2,a}ab_2aK_{2,a}aK_{3,a}aK_{2,a}ad) =$$

$$= aK_{1,a}aK_{2,a}aK_{1,a}aK_{3,a}aK_{1,a}aK_{2,a}aK_{1,a}ab_1aK_{1,a}aK_{2,a}aK_{1,a}aK_{3,a}aK_{1,a}$$

$$aK_{2,a}aK_{1,a}ab_2aK_{1,a}aK_{2,a}aK_{1,a}aK_{3,a}aK_{1,a}aK_{2,a}aK_{1,a}ad.$$

After we replace d by $x_1 aK_1 ax_2$, write down the word g'' in two lines and omit the 13 columns consisting of two a's we get

| $K_{1,a}$ | $K_{2,a}$ | $K_{1,a}$ | $K_{3,a}$ | $K_{1,a}$ | $K_{2,a}$ | $K_{1,a}$ | b_1 | $K_{1,a}$ | $K_{2,a}$ | $K_{1,a}$ | $K_{3,a}$ | $K_{1,a}$ |
| $K_{2,a}$ | $K_{1,a}$ | b_2 | $K_{1,a}$ | $K_{2,a}$ | $K_{1,a}$ | $K_{3,a}$ | $K_{1,a}$ | $K_{2,a}$ | $K_{1,a}$ | x_1 | K_1 | x_2 |

and the entries in the columns are as described in the proof of the theorem.

If we want to be more specific, assume that we are dealing with a Grigorchuk 2-group defined by a sequence that starts with $\omega = 0120\ldots$ and $g = abacad$. Then $K_1 = \{1, d\}$, $K_{1,a} = \{b, c\}$, $K_{2,a} = \{b, d\}$, $K_{3,a} = \{c, d\}$, a possible choice for x_1 and x_2 is $x_1 = b$, $x_2 = c$ and an instance of g'' is the square of

$$g' = ababacacababacababababadac.$$

It appears the theorem has a lot of assumptions, but all of them (except for the existence of r) are satisfied, for example, by any spinal group with $q = 2$.

Let us point out that the theorem shows that there are regular spinal groups with at least linear degree of period growth (those defined by ω as in the theorem with $r = 2$). Moreover, uncountably many examples can be easily found among the Grigorchuk 2-groups (any Grigorchuk 2-group defined by $0 * 0 * 0 * 0 * 0 * \ldots$ where the $*$'s represent arbitrary letters in $\{1, 2\}$).

8. Final Remarks, Open Questions and Directions

It is noticeable that most of the conditions (like r-homogeneous or r-factorable) put on the defining sequences ω throughout the text require appearances of some homomorphisms or subsequences of homomorphisms in a regular fashion with a frequency that could be described and bounded uniformly by the number r. We might note here that any initial segment of ω has no influence on the asymptotics of the growth functions in spinal groups, so that if the desired nice behaviour of ω begins with a little bit of delay we can still use it.

More generally, in some cases we could relax the "uniformly bounded by r" type of conditions to limit conditions that describe the density of appearances of the homomorphisms or subsequences with the desired property.

In the authors' opinion, no spinal group can be finitely presented. The residual finiteness immediately implies this for the spinal groups with non-solvable word problem (uncountably many of them). The solvable word problem case is more involved and interesting and it can probably be handled in a way similar to the way Grigorchuk 2-groups were treated in [Gri84].

This would not come as a surprise if one showed that all spinal groups are branch and just infinite. See [Gri00] for definitions and for more constructions of groups of similar flavor.

An interesting direction in the investigation of the growth problems is introduced by Yuriĭ Leonov in [Leo99] where he connects explicitly the word and period growth of some Grigorchuk 2-groups. For example, Leonov proves that if $\gamma(n) \precsim e^{n^\alpha}$, where $0 < \alpha < 1$, holds for the degree of growth of the first Grigorchuk group, then $\pi(n) \precsim n^{3\alpha}$ holds for the degree of period growth. It would be interesting to describe connections of a similar type in the more general setting of the present paper.

References

[Bar98]   Laurent Bartholdi, *The growth of Grigorchuk's torsion group*, Internat. Math. Res. Notices **20** (1998), 1049–1054.

[Bar00a] Laurent Bartholdi, *A class of groups acting on rooted trees*, unpublished, 2000.

[Bar00b] Laurent Bartholdi, *Lower bounds on the growth of Grigorchuk's torsion group*, to appear in Internat. J. Algebra Comput., 2000.

[Bas72]   Hyman Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614.

[BG99]   Laurent Bartholdi and Rostislav I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, submitted, 1999.

[BG00]   Laurent Bartholdi and Rostislav I. Grigorchuk, *Lie methods in growth of groups and groups of finite width*, Computational and Geometric Aspects of Modern Algebra (Michael Atkinson et al., ed.), London Math. Soc. Lect. Note Ser., vol. 275, Cambridge Univ. Press, Cambridge, 2000, pp. 1–27.

[Bri98]   Matthew G. Brin, *Groups acting on 1-dimensional spaces*, notes for a course at SUNY-B, Bing-
           hamton; available at `http://math.binghamton.edu/matt/index.html`, 1998.

[FG91]   Jacek Fabrykowski and Narain D. Gupta, *On groups with sub-exponential growth functions. II*, J.
           Indian Math. Soc. (N.S.) **56** (1991), no. 1-4, 217–228.

[GC71]   Ferenc Gécseg and Béla Csákány, *Algebraic theory of automata*, Akademiami Kiado, Budapest,
           1971.

[GH90]   Étienne Ghys and Pierre de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*,
           Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990, Papers from the
           Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Gri80]   Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen.
           14 (1980), no. 1, 53–54, English translation: Functional Anal. Appl. **14** (1980), 41–43.

[Gri83]   Rostislav I. Grigorchuk, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR **271**
           (1983), no. 1, 30–33.

[Gri84]   Rostislav I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant
           means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985, English translation: Math.
           USSR-Izv. **25** (1985), no. 2, 259–300.

[Gri85]   Rostislav I. Grigorchuk, *Degrees of growth of p-groups and torsion-free groups*, Mat. Sb. (N.S.)
           126(168) (1985), no. 2, 194–214, 286.

[Gri89]   Rostislav I. Grigorchuk, *On the Hilbert-Poincaré series of graded algebras that are associated with
           groups*, Mat. Sb. **180** (1989), no. 2, 207–225, 304, English translation: Math. USSR-Sb. **66** (1990),
           no. 1, 211–229.

[Gri99]   Rostislav I. Grigorchuk, *On the system of defining relations and the Schur multiplier of periodic
           groups generated by finite automata*, Groups St. Andrews 1997 in Bath, I, Cambridge Univ. Press,
           Cambridge, 1999, pp. 290–317.

[Gri00]   Rostislav I. Grigorchuk, *Just infinite branched groups*, Horizons in Profinite Groups (Dan Segal,
           Markus P. F. du Sautoy, and Aner Shalev, eds.), Birkhaüser, Basel, 2000, pp. 121–179.

[Gro81]   Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci.
           Publ. Math. (1981), no. 53, 53–73.

[GS95]   Ira M. Gessel and Richard P. Stanley, *Handbook of combinatorics*, ch. Algebraic Enumeration,
           pp. 1021–1061, Elsevier, Amsterdam, 1995.

[Gui70]   Yves Guivarc'h, *Groupes de Lie à croissance polynomiale*, C. R. Acad. Sci. Paris Sér. A-B **271**
           (1970), A237–A239.

[Har00]   Pierre de la Harpe, *Topics in geometric group theory*, University of Chicago Press, 2000.

[Leo98]   Yuriĭ G. Leonov, *On growth function for some torsion residually finite groups*, International Con-
           ference dedicated to the 90th Anniversary of L.S.Pontryagin (Moscow), vol. Algebra, Steklov
           Mathematical Institute, September 1998, pp. 36–38.

[Leo99]   Yuriĭ G. Leonov, *On precisement of estimation of periods' growth for Grigorchuk's 2-groups*,
           unpublished, 1999.

[Mil68]   John W. Milnor, *Problem 5603*, Amer. Math. Monthly **75** (1968), 685–686.

[MP99]   Roman Muchnik and Igor Pak, *On growth of Grigorchuk groups*, preprint, 1999.

[Roz86]   Alexander V. Rozhkov, К Теории Групп Алешинского Типа *(russian)*, Mat. Zametki **40** (1986), no. 5, 572–589.

[Tit72]   Jacques Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.